AN ALGEBRA FOR SECONDARY SCHOOLS

BY

E. R. HEDRICK

PROFESSOR OF MATHEMATICS
THE UNIVERSITY OF MISSOURI

6

NEW YORK ·:· CINCINNATI ·:· CHICAGO
AMERICAN BOOK COMPANY

hedrick's algebra.
W. P. I

To

MY SISTER AND TEACHER

MRS. ERNST VOSS *NÉE* AURIE VAIL HEDRICK

THIS BOOK IS DEDICATED

IN GRATEFUL RECOGNITION OF HER LONG

DEVOTION AND ASSISTANCE

PREFACE

THE presentation of a new text on elementary algebra to the public demands justification. It is the author's earnest hope that this book will fill a need felt by many teachers for a book that is at once thoroughly modern, yet conservative of what was good in the older text-books.

The drill afforded by the older texts will not be found wanting. The exercises in all instances are numerous, especially in those topics where drill work is needed to develop technique. It is *strongly recommended that not all the examples in all the lists be solved* on first reading, for the lists have been prepared with a view to reviews, for which some of the exercises should be saved.

Omissions from the text, as well as from the exercises, are to be encouraged. The inclination to do everything between the covers of a book is one of the vicious tendencies of the days when teachers were drillmasters only. Rather let what is done be done with extreme thoroughness. It is probably as valuable a fact as any other which the student may learn that not all of algebra is contained in this or any other book whatsoever; he should be left with the distinct impression that there are many things which he has yet to learn, — among them, perhaps, some of the topics in this book.

The distinctive features of the book are matters of detail. No one favorite principle or prejudice has guided the author, unless it be a desire to speak the *truth*, which is often found an inconvenience. The whole truth is

occasionally withheld, in view of the natural limitations of the scope of the book.

It has also been an aim to meet the saner views — both radical and conservative — which have been expressed in recent Reports of committees throughout the country, on the teaching of elementary algebra.

Thus, several topics traditionally given, and still required by certain colleges for entrance, are placed in the Appendix rather than in the body of the book. Among others, the Euclidian process for finding the H. C. F., the few fragmentary methods for solving special simultaneous quadratics, an explicit treatment of imaginaries, and the theory of limits and infinite series, are relegated to the Appendix, in the belief that none of these topics deserve a place in the usual high school course except for special cause. In the last two topics mentioned, the traditional treatments are by no means free from errors, and this fact alone would indicate that they are neither suited to the child's intelligence, nor particularly valuable to him.

Extreme rigor of proof is not exacted, and the psychological advantage of *conviction* as compared with *proof* is recognized; but frank statements are made whenever the proofs are not complete.

The language of the book is purposely simple, frank, and conversational. A special effort is made to impart to the student the ability to elucidate English in exercises, and to translate English into formulas — the principal advantage in the algebraic notation.

Graphical illustrations are treated as a normal part of algebraic knowledge. They are used, whenever they are valuable, without extensive discussion and without ostentatious use of needless nomenclature. It will be found that no complicated curves are used in exercises for the student. Teachers who realize the tremendous value

of graphical work, both in everyday affairs and in the study of mathematics and science, will welcome the *consistent* use of this tool, probably the most useful of algebraic tools for actual problems of life and science. The early introduction of the simpler notions will not alarm a teacher who has *tried* these ideas with very young students.

In the practical exercises, the aim has been to eliminate artificial problems of the most extreme type — those in which the data could not conceivably be known before the answers were known. Such problems tend to destroy interest and sympathy. Problems intended for drill work may be artificial, if there is no pretense of clothing them in hypocritically practical language.

While it meets the entrance requirements of American colleges and universities generally, this book is written essentially for those for whom the high school course is to be the last. Especially for such students, it would be deplorable to omit the principal features of the body of the book in favor of those in the Appendix. Such students are in a majority in most high schools.

An unusually large number of persons have assisted me in various ways. I am indebted to Mr. W. A. Hurwitz, for the preparation of a large number of the exercises; to Professor L. D. Ames, for the preparation of portions of the text; to Professor O. D. Kellogg, Superintendent J. M. Greenwood (Kansas City), Mr. W. L. Jordan (Des Moines), and others, for suggestions upon the manuscript; to Dr. Louis Ingold, for checking all answers. For suggestions upon the proofs, I would especially thank Professor W. F. Osgood (Harvard), Professor James Pierpont (Yale), Professor F. N. Cole (Columbia), Professor C. A. Waldo (Purdue), Professor M. A. Bussewitz (Milwaukee Normal School), Professor D. F.

Campbell (Armour Institute), Professor E. A. Lyman (Michigan Normal School), Professor H. C. Harvey (Kirksville Normal School), Professor Ira S. Condit (Iowa Normal School), Professor A. H. Wilson (Alabama Polytechnic Institute), Messrs. E. D. Phillips, E. M. Bainter, and A. A. Dodd (Kansas City), Mr. A. M. Allison (Sioux City, Ia.), Mr. Lewis Omer (Oak Park, Ill.), Mr. R. H. Jordan (St. Joseph). The book owes much to the suggestions of these and other persons. To all who have assisted me, I would acknowledge here my indebtedness.

E. R. HEDRICK.

CONTENTS

ALGEBRA FOR SECONDARY SCHOOLS

CHAPTER I. INTRODUCTION

PART I. NUMBERS AND SIGNS

1. Algebra. The word *algebra* is used as a convenient name for the continuation of arithmetic; there is no definite line between the two subjects, but symbols are used more freely in algebra than in elementary arithmetic.

2. Numbers. The first numbers we learn are the **integers**, 1, 2, 3, 4, etc. The students know also another kind of numbers, **fractions**, which are useful in measuring quantities. Later we shall extend these ideas.

3. Signs and Marks. Various signs and marks, sometimes called **symbols**, or **characters**, are used as in arithmetic, and new ones are introduced as they are needed. A table of these is found at the end of this book (p. 407).

Addition, *subtraction*, *multiplication*, and *division* are indicated by the signs, $+$, $-$, $\times$, $\div$, respectively, as in elementary arithmetic.

Thus, $3 + 2$, $3 - 2$, 3×2, $3 \div 2$, are read "3 plus 2," "3 minus 2," "3 multiplied by 2," and "3 divided by 2," respectively. The *product* 3×2 is also often written $3 \cdot 2$, the dot $\cdot$ being a simplified cross, $\times$; and we may say "3 times 2" instead of "3 multiplied by 2." The quotient $3 \div 2$ is sometimes written $3:2$, also $\frac{3}{2}$, also $3/2$; the form $3:2$ is often called the ratio form; the form $\frac{3}{2}$ (as well as the form $3/2$) is often called the *fraction* form; but they all mean the same thing. The form $\frac{3}{2}$ may also be read "3 over 2."

One important sign is $=$, read "equals," or "is equal to." A statement that one thing is equal to another is called an **equation** or an **equality**; *e.g.* $2+4=6$. Two numbers are **equal** if, and only if, they are the same number, though they may be written differently. Thus, $2+4$ is the same number as 6, but it is written differently.

An equality of two fractions (or ratios) is called a **proportion**. Thus, $\frac{2}{4}=\frac{3}{6}$ (or $2:4=3:6$) is a proportion.

The **radical sign** $\sqrt{\ }$ is used to indicate square root; thus $\sqrt{16}=4$. For further uses of this sign, see p. 9.

4. Abbreviations. In arithmetic we use such abbreviations as ft. for 1 foot, in. for 1 inch, hr. for 1 hour, etc.; and we write 36 in. = 3 ft., etc. In algebra we abbreviate still more. Thus, we may as well write merely f for one foot and i for one inch, then $36\,i=3\,f$. Such abbreviations often simplify problems and shorten calculations; they make general statements possible. Thus, instead of saying

$$\text{interest} = (\text{rate}) \times (\text{principal}) \times (\text{time in years}),$$

we may say

$$i = r \times p \times t$$

with the same advantage that we gain in writing Mo. for Missouri; namely, *we save time and space.*

There is nothing mysterious or difficult about such abbreviations if they are once explained. It should be noticed that the same letter may be used for different things *in different problems*; thus i was used to mean one inch in one connection above, and again to mean interest. Such abbreviations are simply temporary, for a single problem; the student will not be confused if he clearly understands what each letter means. It is *very important*, however, to avoid using the same letters for two different things *in the same problem.*

In using letters as above, the sign of multiplication is often omitted; thus rpt means $r \times p \times t$, and $5\,pr$ means $5 \times p \times r$; but it is not safe to omit the sign when two factors are ordinary Arabic numbers, since, for example,

52 means $50 + 2$, not 5×2. However, this may be done in combining numbers and letters, as in arithmetic, when we say that if in. means 1 inch, 3 in. means 3 inches.

EXERCISES I: CHAPTER I

[Tables of symbols, weights, measures, etc., will be found at the back of the book.]

1. Express $4f + 3i$ in inches if f means one foot and i means one inch.

2. Express $2h + 4m - 14s$ in seconds if h means one hour, m one minute, and s one second.

3. If b denotes one bushel, p one peck, and q one quart, express $3b - p - 2q$ in quarts. Also express $3b - p - 2q$ in pecks; in bushels.

4. If t stands for one tenth and h for one hundredth, express $1 - 3t + 5h$ in per cent.

Express 67 % in terms of t and h.

5. Let c denote one cubic centimeter, i one cubic inch, and f one cubic foot. Express $\frac{1}{12}f - 132\,i$ in terms of c, if we let i equal $16\,c$ (which is very nearly correct).

6. Express $2m + 30y + 3f$ in terms of f if $m = 5280f$ and $y = 3f$.

[Note that figures given are actually applicable to the case in which f denotes one foot, y one yard, and m one mile.]

7. If $g = 4q$, express $2g + 3q$ in terms of q; in terms of g.

[Can you mention a case of measurement to which this problem would be applicable?]

8. If y denotes one yard and i one inch, express $m = y + 3\frac{3}{8}i$ in terms of i. What unit of length does m denote?

9. If x denotes a single thing, d a dozen, and m a gross, express $2m + 11d - 6x$ in terms of d.

10. Express $a - 17b + 8c - 50d$ in terms of d if $a = 24b$, $b = 60c$, $c = 60d$. To what measures will this apply?

11. What is the value of $4a + 3b - 3c$ if $a = 2$, $b = 5$, $c = 3$?

12. What is the value of $6xt - 3y - 5t$ if $x = 3$, $y = 0$, $t = 2$?

13. Express $4m - 3n + 10p$ in terms of n if $m = 2n$, $n = 5p$. What is the value of the result if $n = 52$? if $n = 32$? if $n = 0$?

14. What is the value of $\dfrac{2ax + by + c}{3x - b}$ if $a = 3$, $b = 4$, $c = 6$, $x = 5$, $y = 2$?

15. If $A = 5a$, $B = 3b$, express $\dfrac{A + B}{Ab - Ba}$ in terms of a and b. Find the value of this result if $a = 2$ and $b = 4$.

16. If x denotes the area of a square, and y the length of a side, then $x = y \cdot y$. What is the area of a square whose side is 4 feet long? (The product $y \cdot y$ is often written [illegible] is called the *square* of y because $y \cdot y$ is the area of the square.)

17. Express $\dfrac{x + y}{1 - xy}$ in terms of x alone if $y = 2x$. Then find its value if $x = \frac{1}{2}$.

18. What is the area of a rectangle whose length is a and whose breadth is b? What is the area of a rectangle 5 inches long and 3 inches wide?

19. Let m denote the side of a cube, n its volume. Express n in terms of m. If $m = 2$ feet, what is n? If $n = 27$ cubic inches, what is m?

20. Let the dimensions of a room, measured in feet, be denoted by w (the width), l (the length), and h (the height of the ceiling); and let c be the cost (in cents) per square yard of plastering. How much will it cost to plaster the walls and ceiling of the room, neglecting doors and windows?

What will the cost be if $l = 16$ feet, $w = 15$ feet, $h = 10$ feet, and $c = 15$ cents?

5. Problem Solving. *Abbreviations* are especially useful in the solution of problems.

Ex. 1. Find a number such that the sum of that number and half the number is 18.

SOLUTION (not abbreviated)	SOLUTION (abbreviated)
Consider *the number to be found.*	Let n = *the number to be found.*
Then (that number) + $\frac{1}{2}$ (that number) = 18,	Then $n + \frac{1}{2}n = 18$,
or, $\frac{3}{2}$ (that number) = 18,	or, $\frac{3}{2}n = 18$,
or, $\frac{1}{2}$ (that number) = 6,	or, $\frac{1}{2}n = 6$,
or, (that number) = 12 (answer).	or, $n = 12$.
Check:	*Check:*
$12 + \frac{1}{2} \cdot 12 = 18$. (Correct.)	$12 + \frac{1}{2} \cdot 12 = 18$. (Correct.)

A **check** is any process for testing an answer. If, as here, the answer can be tried directly in the given problem, the check is **complete**; a complete check shows that the answer is surely correct.

[The teacher should explain carefully the check on the answer in this problem, and the students should be required to check their answers in all cases.]

Ex. 2. A merchant sold tea for 35 cents per pound. What was the cost to him if he made 25 % profit on the cost?

SOLUTION (not abbreviated)	SOLUTION (abbreviated)
Consider *the cost to merchant.*	Let c = *the cost to merchant.*
Then the profit to merchant is 25 % of cost to merchant,	Then profit to merchant = 25 % c.
or $\frac{1}{4}$ of cost to merchant.	Then profit to merchant = $\frac{1}{4}c$.
Hence, the total price is the cost to merchant added to $\frac{1}{4}$ of cost to merchant,	Hence total price is $c + \frac{1}{4}c$.
or $\frac{5}{4}$ of cost to merchant.	Hence total price = $\frac{5}{4}c$.
Then 35 cents is $\frac{5}{4}$ of cost to merchant,	Then 35 cents = $\frac{5}{4}c$,
or 28 cents is the cost to merchant.	28 cents = c.
Check: $28 + \frac{1}{4} \cdot 28 = 35$.	*Check:* $28 + \frac{1}{4} \cdot 28 = 35$.

EXERCISES II: CHAPTER I

[Check each result as indicated in the preceding examples.]

1. If the sum of three times a certain number and half that number is 14, what is the number?

[Let the students invent and propose to each other such problems.]

2. What number yields a remainder 5 if diminished by the sum of one half of itself and one third of itself?

3. What must be the amount of money invested in an enterprise yielding 25% profit, in order that the money in hand at its conclusion may be $1000?

4. A man attempted to charge 8% interest on money. Being able to collect only the legal rate (6%) he made $20 less than he expected. What was the amount of money loaned?

5. A merchant buys butter for 30¢ a pound and sells it for 36¢ a pound. What is his per cent of profit on the cost?

6. A fruit dealer sold oranges at 5 cents apiece or 50 cents per dozen. He found that he received $54.00 for 100 dozen. How many were sold singly and how many by the dozen?

7. What must be the per cent of profit on an investment if $525 is to produce $600?

8. A shoe dealer buys 100 pairs of shoes at $2.00 each and sells 75 pairs at $2.50 each. To sell the remaining 25 pairs he marks them down so as to make 20% profit on the whole. What is the price per pair that will give this result?

9. I propose to a friend the following puzzle: "Think of any number, add 5 to it, multiply the result by 3, and subtract 4." He gives the result as 17. What was his number?

10. I have an opportunity to lend money at 5% simple interest for a period of 8 years. How much must I lend in order that the amount may be $3500?

11. How large a cubical box may be covered with 24 square inches of paper?

PART II. PRELIMINARY DEFINITIONS

6. Expressions. All single groups of numbers and characters of the kind already used are called **expressions**; in order to deserve this name, the group of numbers and characters must have some meaning.

Thus, $3f + 2i$ has a meaning if f means 1 ft. and i means 1 in.; then $3f + 2i$ is an expression.

Again, the group of characters $p + rpt$ is an expression, although the meaning is not wholly clear. It is at least clear that some number p is to be increased by the product of three quantities, r, p, and t. The meaning becomes clearer if we are told that p means the principal, r the rate, and t the time in years in a certain interest problem; then the above expression clearly means the amount. The meaning becomes still clearer when the numerical values of r, p, and t are given, say $r = 5\%$, $p = \$125$, and $t = 3$ years. These various stages in clearness do not contradict the fact that $p + rpt$ has in itself (without any further explanation) a certain meaning as expressed above.

7. Terms. An expression may contain one or more + or − signs, which separate it into parts; these parts are called the **terms** of the expression.

Thus in $p + rpt$ the terms are p and rpt. In $2ax + \frac{5y+2}{3} + 4c$ the terms are $2ax$ and $\frac{5y+2}{3}$ and $4c$. To be sure, the term $\frac{5y+2}{3}$ itself contains a + sign, but it is not separated into parts by that sign as it stands. On the whole, the word *term* is used rather loosely, the intention being to distinguish those parts which stand in rather compact groups as compared with the rest.

The terms of an expression are calculated separately, and these results are added or subtracted in their order. Thus, as a general rule, the multiplications and divisions are carried out first, after which the terms thus formed are added or subtracted. If anything else is intended, *parentheses* are used to show that the expression inside the parentheses is to be calculated first. Thus, the expression $2ax + m(5y + 2) + 4c$ has three terms: $2ax$, $m(5y + 2)$, and $4c$; the term $m(5y + 2)$ means m times the sum of $5y$ and 2. See also § 11, p. 10.

To distinguish expressions the special names **monomial, binomial**, and **trinomial** are often used for expressions that have *one term*, *two terms*, or *three terms*, respectively.

Thus rpt and $\frac{5y+2}{3}$ are monomials, $p + rpt$ is a binomial, and $2ax + \frac{5y+2}{3} - 4c$ is a trinomial, etc.

When we wish to refer to a complicated expression of more than three terms, we may simply call it an **expression**. The word *polynomial* may also be used in certain simple cases defined in § 9 below.

8. Factors. When several numbers or expressions are multiplied together, any one of them is called a **factor** and the result is called the **product.**

Thus the factors of the product rpt are r, p, and t.

Any factor is called the **coefficient** of the rest of the product; usually, however, the word *coefficient* is understood to be that factor which is represented by Arabic numerals, or which is supposed to be a known number, unless the coefficient of a special part is required.

Thus, in $2axy$ the coefficient is 2 unless a specific part is mentioned, but if we are asked for the coefficient of xy, the answer is $2a$; the coefficient of $2ax$ is y, and so on. *Coefficient* is but another name for *multiplier*. Thus, in $2axy$, 2 is the multiplier of axy and $2a$ is the multiplier of xy; but *coefficient* is the word generally used.

If the coefficient is 1, it is not written, since multiplying a number by 1 does not change it. Thus, in axy the coefficient is 1 unless the coefficient of a special part is required.

Terms that are *precisely the same or that differ only in their coefficients* are called **similar terms** or **like terms.** Thus, $2x$ and $3x$ are similar; $3m^2n$ and $4m^2n$ are similar. Terms that are not similar are called **dissimilar** or **unlike.**

9. Powers. The product of two equal factors is called the **square** of that factor; it is indicated by a small figure 2 at the upper right-hand corner. Thus, $x \cdot x = x^2$.

The product of three equal factors is called the **cube** of that factor. Thus, $x \cdot x \cdot x = x^3$.

The product of any number of equal factors is called a **positive integral power** or a **simple power** of that factor; it is indicated by writing the number of the factors at the upper right-hand corner of the factor.

Thus, we write $x \cdot x \cdot x \cdot x = x^4$; $x \cdot x \cdot x \cdot x \cdot x = x^5$; etc.

The number of factors is called the **exponent** of the power. Thus, in x^4 the exponent is 4, and x^4 is called *the fourth power of* x, or simply x *fourth power*. We shall later extend these definitions of power and exponent so as to give a meaning to such forms as $x^{\frac{2}{3}}$, x^{-2}, x^0, etc., which are at present meaningless to the student. See pp. 286, 296.

Since any number x may be considered as taken once as a factor to make itself, x^1 *means the same as* x. Hence, it is useless to write the exponent 1, and when no exponent is written, 1 is understood.

A **polynomial** in certain given letters is an expression whose terms, if they contain one of those letters at all, contain it as a factor which is a simple power.*

10. Roots. If a number is the product of two equal factors, one of the equal factors is called the **square root** of the number; if a number is the product of three equal factors, one of the equal factors is called the **cube root.** In general, if a quantity is the product of a number of equal factors, one of them is called a **root** of that quantity; and the number of the factors is called the **index** of the root.

Thus, 4 has two equal factors, 2 and 2. Hence, the square root of 4 is 2, written $\sqrt{4} = 2$; likewise, since $3 \cdot 3 \cdot 3 = 27$, the cube root of 27 is 3, written $\sqrt[3]{27} = 3$. Again, since $a \cdot a = a^2$, $\sqrt{a^2} = a$. So also, $\sqrt[3]{a^3} = a$; $\sqrt[4]{a^4} = a$, etc. The index is always written at the upper left-hand corner of the sign $\sqrt{\ }$, except for square roots, which are indicated by $\sqrt{\ }$ instead of $\sqrt[2]{\ }$. These ideas will be extended later.

---

*A broader use of *polynomial* is common in many text-books, but it is not good usage. Standard mathematical works other than text-books agree on the definition used above. The word is not used extensively in this book until after p. 88, where a more detailed explanation is given.

11. Parentheses. When we wish to group terms together, we use ordinary **parentheses** (see § 7, p. 7). Thus, $(a+b)^2$ means the square of the expression $a+b$.

In order to avoid confusion when one pair of parentheses occurs within another, we use different forms, as follows: [ ], called *brackets;* { }, called *braces;* and $\overline{\quad\quad}$, called the *vinculum.* These names are convenient to distinguish the shapes, but they are all used alike, and they are all called parentheses except when some confusion would result.

Thus $(2+3)\cdot 4$ means $5\cdot 4$, or 20; $(a+b)c$ means the sum of a and b multiplied by c; $2\sqrt{a+b}$ means twice the square root of the sum $(a+b)$. We may now use more complicated expressions, the convenience of which will be seen in the next chapter. For instance,

$$2(a+b)(c+d)+(a+b)^2$$

means twice the product of (a plus b) and (c plus d) added to the square of (a plus b). If the letters used mean certain numbers, say $a=2$, $b=1$, $c=3$, $d=4$, then $a+b=3$, $c+d=7$, and

$$2(a+b)(c+d)+(a+b)^2=2\cdot 3\cdot 7+3^2=42+9=51.$$

EXERCISES III: CHAPTER I

Find the value of:

1. $2a^3$ if $a=2$; if $a=3$; if $a=5$.

2. $4a^2b^3$ if $a=2$, $b=3$; if $a=1$, $b=2$; if $a=5$, $b=1$.

3. $\sqrt{pr}$ if $p=2$, $r=8$; if $p=3$, $r=12$; if $p=18$, $r=8$.

4. $x^3+4-4x-x^2$ if $x=1$; if $x=2$; if $x=3$; if $x=4$.

5. $\sqrt{5a^3b^2}+4a^2(b+c)$ if $a=5$, $b=2$, $c=3$; if $a=20$, $b=1$, $c=2$.

6. Find the value of $\dfrac{x\sqrt{1-y^2}-y\sqrt{1-x^2}}{xy+\sqrt{1-x^2}\cdot\sqrt{1-y^2}}$ if $x=\frac{4}{5}$, $y=\frac{3}{5}$

7. $\dfrac{b\sqrt{x^2-y^2}+y\sqrt{a^2-b^2}}{\sqrt{(x^2-y^2)(a^2-b^2)}-by}$ if $a=5$, $b=3$, $x=13$, $y=5$.

8. 3^n if $n=2$; if $n=3$; if $n=1$.

9. x^y if $x=2,\ y=3$; if $x=4,\ y=1$; if $x=1,\ y=2$.

10. $\sqrt[b]{\sqrt[c]{a}}$ if $a=64,\ b=3,\ c=2$; if $a=256,\ b=4,\ c=2$.

Express:

11. $(2f+3i)^2$ in square inches if $f=$ one foot and $i=$ one inch.

12. $(2t+u)^2$ in units if t denotes one ten and u one unit. Then write the result in terms of t^2, t, and u.

Show that:

13. $(x+y)^2=x^2+2xy+y^2$ if $x=3,\ y=2$; if $x=5,\ y=1$; if $x=\frac{1}{2},\ y=\frac{3}{2}$.

14. $(n^2+1)^2=(n^2-1)^2+(2n)^2$ if $n=1$; if $n=2$; if $n=3$; if $n=4$.

15. $\dfrac{a^3+b^3}{a+b}=a^2-ab+b^2$ if $a=3,\ b=2$; if $a=5,\ b=1$.

16. $x^2-(r+s)x+rs=(x-r)(x-s)$ if $x=4,\ r=1,\ s=2$; if $x=\frac{5}{2},\ r=\frac{1}{2},\ s=1$; if $x=3,\ r=3,\ s=1$.

17. $(p^2+q^2)(p+q)(p-q)=p^4-q^4$ if $p=2,\ q=1$; if $p=3,\ q=2$.

18. In Ex. 1 name the coefficient of a^3; the exponent of a.

19. In Ex. 2 name the coefficient of a^2b^3; of a^2; of b^3; the exponent of a; of b.

20. In Ex. 4 name the coefficient and the exponent in each term.

21. In Ex. 5 what is the coefficient of a^2 in the second term?

22. Name the exponent of 3 in Ex. 8; of x in Ex. 9. Name the coefficients in the result of Ex. 12.

23. Give some factors of the expressions found in Ex. 1; in Ex. 2.

SUMMARY OF CHAPTER I: INTRODUCTION TO ALGEBRA
pp. 1–11

CHAPTER II. MEASUREMENT AND AIDS IN EXPRESSION

PART I. MEASUREMENT OF SIMPLE QUANTITIES. NEGATIVE NUMBERS

12. Simple Measurement. In measuring lengths we are used to marking off, on a straight line, equal distances which represent equal units, as in Fig. 1. It is often convenient to sub divide these units, either into tenths or into some other number of parts. Fig. 2 represents part of a ruler divided into inches and eighths of an inch.

Fig. 1.

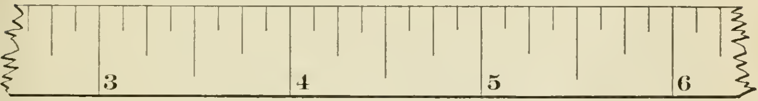


Fig. 2.

Fig. 3 represents part of a meter stick ruled in centimeters and millimeters.

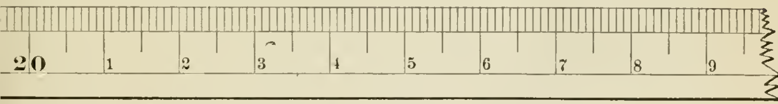


Fig. 3.

13. Thermometers. Other kinds of quantity are often measured similarly. Thus, a thermometer is marked off in divisions, each of which corresponds to one degree.

The starting point, which we call zero, on a Fahrenheit thermometer, was put where it is because the inventor could not artificially cause a lower temperature by means of his snow and salt mixtures.

14. Negative Numbers. The thermometer may fall even *below* zero; if it does, we ordinarily say that the temperature is so many degrees "below zero."

There is an abbreviated way of saying the same thing. Let us suppose the temperature is 60° at noon and falls 20° by night. Then at night the temperature is $60° - 20° = 40°$. But if it is 0° at noon and falls 20° by night, then at night we say it is $0° - 20°$, thereby meaning 20° *below* zero. Thus, "20° *below zero*" is often written simply "$-20°$," the 0° being omitted from the expression $0° - 20°$ because it is useless.

This system works very well. For example, if it was 30° above zero and has fallen 40°, meanwhile it must have been exactly zero at some time, just after it had fallen 30°. It then must have fallen 10° more, making 10° *below zero* or $-10°$. That is, $30° - 40° = -10°$. This result is very convenient, for $30° - 40°$ *ought*, as above, to be equal to $30° - 30° - 10°$ or $0° - 10°$ or $-10°$.

This abbreviation consists, then, in writing "$-10°$" for "10° *below zero*," etc.; in general $-n° = n°$ *below zero*.

Similar quantities arise frequently. For example, if a man has \$40 and spends \$15, we say he has \$40 – \$15, or \$25. But if he has \$40 and spends \$50, we say he is *in debt* \$10. It is simpler to express this as "–\$10."

Again, if a weight of 20 pounds and a weight of 30 pounds are hung together on a scale, the total weight is 20 pounds + 30 pounds = 50 pounds; but if a weight of 50 pounds is attached to a balloon, the total weight is diminished. We therefore say the balloon weighs *less than zero* and indicate this by writing the minus sign in front of the amount it pulls up. Thus, if a 50-pound weight is reduced to 30 pounds by tying a balloon to it, we say the balloon "*pulls up* by an amount of 20 pounds"; or that "its *weight* is –20 pounds."

Numerous examples of this sort may be given, with pairs of quantities that are exactly the opposites of each other, so that equal amounts of the two kinds counterbalance or destroy each other. Thus *money and debts*, *weights and balloons*, temperatures *above and below zero*,

and so on, are such pairs. One being chosen as the original quantity considered, the contrasting one is denoted by the sign $-$ prefixed to its amount. The contrasting one (denoted by the sign $-$) is called **negative**, the original one being called **positive**.

This arrangement can in any case be reversed. Thus money may be thought of as the *opposite* of a debt; then *to have* \$ 10 $= -$ \$ 10 *in debt*, but we always try to choose the simpler way in any given case. Sometimes it is quite immaterial which quantity is called *positive* and which is called *negative*. Thus, walking east may be regarded as *positive progress* if one really wants to go east. If a man who is lost wanders 10 miles west when he intended to go east, we may say he has gone -10 miles east. In this case, the circumstances determine whether going east or going west is called positive; but if one is called positive, its opposite thereby becomes negative.

It is important to keep the same agreement throughout any one problem, after a choice has been made.

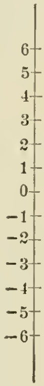


Fig. 4.

15. Representation of Negatives. We may always represent negative quantities just as on a thermometer. Drawing a vertical straight line, we shall agree that upwards is positive, then downwards is negative. The starting point is marked zero. Successive steps upward are marked 1, 2, 3, 4, 5, and so on, the distances between marks being equal. The similar equal spaces below the starting point are marked -1, -2, -3, etc., as on a thermometer. The number **zero** mentioned above is very important.

It is easy to compute on this scale. If we wish to add 4 to 2, for example, we start at 2 and take 4 upward steps, thus reaching 6. If we wish to subtract 3 from 7, we start at 7 and take 3 steps down, thus reaching 4. But we may perform other operations, some of which might seem difficult. Thus we may add 5 to -2 by starting at -2 and taking 5 steps upward, thus reaching 3. Or we may subtract 5 from 3 by starting at 3 and taking 5 steps downward, thus reaching -2. These operations are new to the student.

The line used need not be in any special position; often it is drawn horizontal, and distances to the right are then usually called positive; to the left, negative.

Fig. 5.

Fractions are indicated, as on a ruler, by subdividing the unit steps. Thus $2\frac{1}{2}$ is the point halfway between 2 and 3; $-2\frac{1}{2}$ is the point halfway between -2 and -3; and so on, in a natural manner. A convenient division of the unit is into tenths, as indicated in the figure. We shall discuss this further in Chapter III.

EXERCISES I: CHAPTER II

Add:

1. 5 to -1.

2. 3 to -4.

3. 2 to -2.

4. 7 to -3.

Subtract:

5. 3 from 1.

6. 2 from -2.

7. 4 from 4.

8. 7 from 2.

Locate on a line the points that correspond to the following numbers:

9. $3\frac{1}{2}$. **10.** $2\frac{1}{5}$. **11.** $-1\frac{1}{2}$. **12.** $6\frac{1}{3}$. **13.** -4.3.

Add:

14. $3\frac{1}{2}$ to 4.

15. $3\frac{1}{2}$ to 4.2.

16. 4 to -2.4.

17. 2.3 to -4.2.

Subtract:

18. 5 from 3.2.

19. 3 from $-1\frac{1}{2}$.

20. 4.2 from $3\frac{2}{5}$.

21. 1.5 from $-2\frac{3}{4}$.

PART II. RELATIONS BETWEEN TWO QUANTITIES; GRAPHS

16. Representation of Quantities. Let us suppose that a thermometer is read every hour, and that its reading is put down. We shall have a table similar to that which follows:

	A.M.				NOON	P.M.							
Time of Day	8	9	10	11	12	1	2	3	4	5	6	7	etc.
Temperature	46°	48°	51°	55°	56°	57°	56°	52°	50°	45°	43°	42°	etc.

These numbers may be represented on a double scale on paper marked with lines both ways, *i.e.* horizontally and vertically. The diagram represents such a figure.

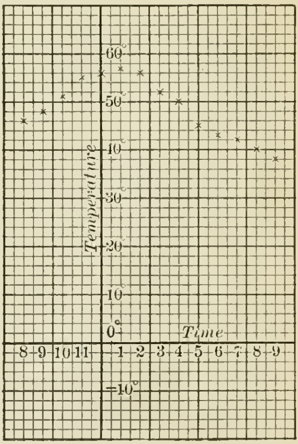


FIG. 6.

The time is measured in this figure along the horizontal line, and the various hours are marked at equal spaces, A.M. being at the left and P.M. at the right of the starting point. The temperatures are measured in the figure vertically as marked. At 8 A.M. the temperature is 46°. We indicate this by a small cross just above "8 A.M." and opposite "46°" in the figure. At 9 A.M. the temperature is 48°; this is indicated by the cross above 9 A.M. and opposite 48°. The various marks indicate the various temperatures at the different hours of the day as given in the table.

In this figure we have represented positive temperatures only, *i.e.* temperatures "above zero." If we take some day on which the temperature "falls below zero," we must use the part of the vertical line marked with minus signs, as shown in the figures on pp. 18 and 19.

On a certain day the temperatures were:

Time of Day	8	9	10	11	12	1	2	3	4	5	6	7	8	9
Temperature	– 10	– 6	0	4	10	12	20	18	16	15	12	10	5	– 2

The diagram that corresponds to the preceding table is shown in Fig. 7:

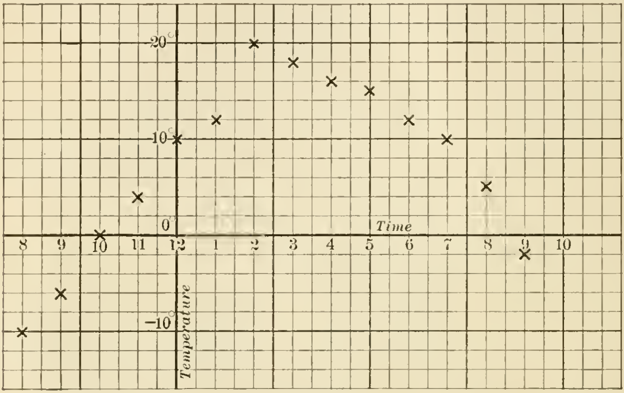


Fig. 7.

If the temperature is read oftener, say every fifteen minutes, a much better figure results. Such a figure is shown in Fig. 8.

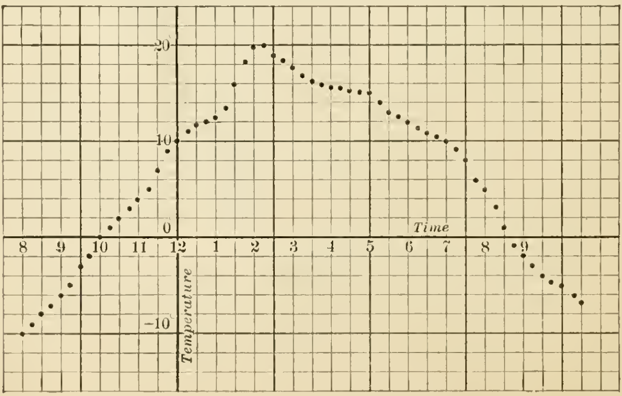


Fig. 8.

Let the student judge as nearly as he can from this figure the temperature at 8.30 A.M., at 11.15 A.M., at 2 P.M., at 9.15 P.M. Notice that this cannot be told accurately, but only approximately. Notice also that the temperature can be estimated at times when it was not really measured. Thus the temperature at 11.40 A.M., though not actually measured, may be judged by the figure to have been about 8°. A smooth curve drawn through the points, as shown in Fig. 9, gives a very accurate idea of what the temperature was at any moment.

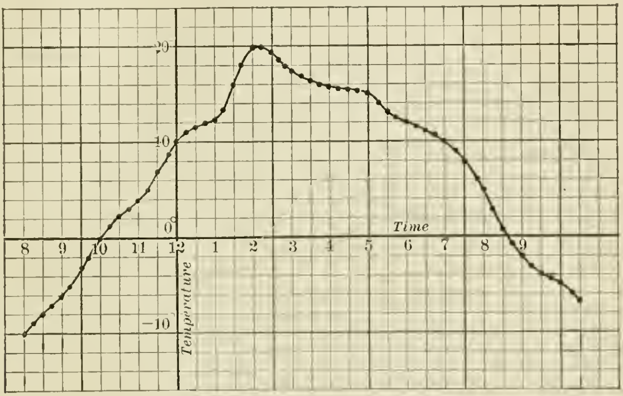


FIG. 9.

The figure may be used also in another way. We may ask, "At what time was the temperature 14°?" By following the horizontal line through the mark "14" till it strikes the curve, we see that the temperature was 14° at *about* 1.20 P.M. and again at *about* 5.15 P.M. It should be noticed that there are two answers.

Let the student select the time of day from the figure when the temperature was 18°; 10°; 0°. Notice that it was never 30° on that day. Let the student actually use a thermometer and draw such a figure as that just given, and let him then show what the temperature must have been (approximately) at some time when he did not measure it. Let him also show (about) when the temperature was (say) 40°.

Such figures as those just drawn are often called **diagrams**, or **graphs**. Drawing such a diagram or graph is called **plotting** it; locating a point is called *plotting* the point.

17. Scales of Prices. Many quantities may be drawn in figures like the preceding. Following are examples:

Ex. 1. If butter is 30 cents per pound, draw a figure to represent the cost of any number of pounds.

We first make a table as shown below:

No. of lb.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	etc.
Price in ¢.	30	60	90	120	150	180										

[Let the student fill in the blank spaces.]

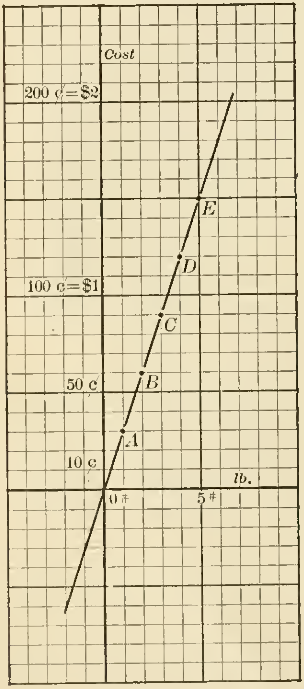


Fig. 10.

Then, representing the number of pounds of butter on a horizontal line, as shown, and the cost on a vertical line, as shown, we have as before a set of points, A, B, C, D, E, $\cdots$. Joining these by a smooth curve, we get a close idea of the cost of any number of pounds, even a fractional number. For example, the cost of $3\frac{1}{2}$ pounds is \$1.05, as the figure indicates. The "curve" in this case is really a straight line; this may happen in any example, but the word *curve* is used at least until we are sure the line is really straight. Most price curves are straight lines. See §§ 18, 19, pp. 23, 24.

Notice that *no butter cost no money.* Thus at the point marked "0" on the horizontal line we should draw the point *at the height* 0.

Notice that *buying* -2 *pounds of butter means selling* 2 *pounds.* The *cost* of -2 pounds is, therefore, *less* than nothing; in fact, a man is, of course, *paid for the butter* if he really *sells* it. Instead of saying that he is *paid* 60 cents for *selling* 2 pounds, we may say that the *cost* is -60 cents if he *buys* -2 pounds. This would be represented in

the figure by a point corresponding to -2 on the horizontal line and to $-60¢$ on the vertical line. With this understanding the curve of prices just above is an unbroken straight line.

EXERCISES II: CHAPTER II

1. Draw a figure to represent the temperature at various times of the day from the following table:

	A.M.				M.	P.M.									
Time . . .	8	9	10	11	12	1	2	3	4	5	6	7	8	9	etc.
Temperature	30.5	31.5	31	30.5	29	28.5	29.5	31	31.5	31	30	29.5	29	28	etc.

2. From the figure in Ex. 1 estimate the temperature at at 11.30 A.M.; at 2.30 P.M.; at 6.15 P.M.

3. At what time of day was the temperature about 30.5°? 29°? Is there more than one answer in each case? Find *all* the answers.

4. At what times (about) during the day did the temperature change from rising to falling? from falling to rising?

5. If the price of tea is 35 cents per pound, draw a figure to show the cost of any number of pounds.

6. From the figures give the cost of $3\frac{1}{2}$ pounds; of $4\frac{1}{2}$ pounds; of $-1\frac{1}{2}$ pounds.

[HINT. — Continue the line below and to the left.]

7. How much tea can be bought for 84 cents? for \$1.96? for $-$ \$1.05? What does the last statement mean?

8. The population of the U.S., as determined by the decennial census, is approximately given in the following table:

Year . .	1790	1800	1810	1820	1830	1840	1850	1860	1870	1880	1890	1900
Millions .	4	5	7	10	13	17	23	31	39	50	63	76

Draw a graph showing the population at various times.

9. Estimate the population in 1805; in 1885; in 1895.

10. When was the population about 15,000,000? 60,000,000? 70,000,000?

11. Represent by a graph the simple interest at 6 % for one year on any amount of money. Choose sums at intervals of \$1000 from – \$5000 to + \$5000.

12. What interest shall I have to pay at 6 % for a year's use of \$2500? \$4500? – \$2500? Interpret the last statement in the light of Ex. 11.

13. What principal yields as interest for one year at 6% \$195? – \$195? – \$105?

14. The number of inhabitants of the United States of school age from 1870 to 1904 is shown in the Fig. 11. What was the number in 1870? 1880? 1885? 1890?

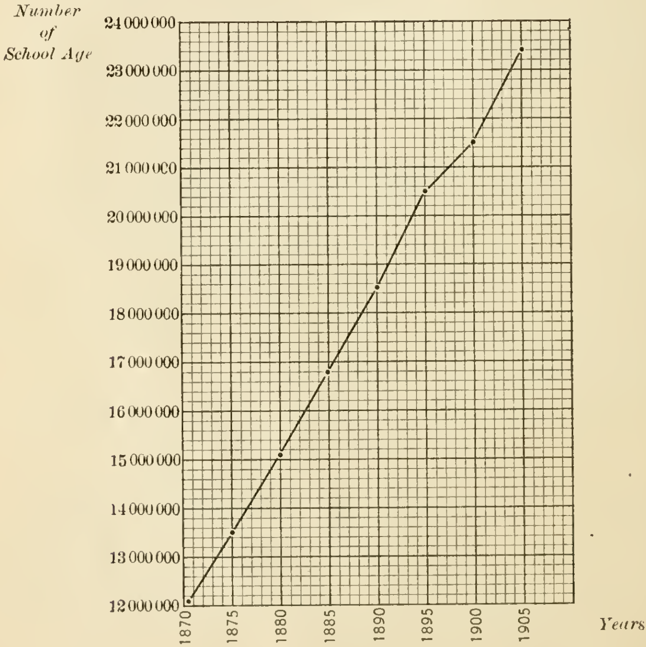


FIG. 11.

15. When was the number of inhabitants of school age about 12,000,000? 14,000,000? 16,000,000? 20,000,000?

16. From the figure construct a complete table of the number of inhabitants of school age from 1870 to 1905 (every year).

17. Draw a straight line graph to represent the corresponding readings of a Centigrade and a Fahrenheit thermometer. (See tables at back of book.)

18. Give the Centigrade temperatures corresponding to Fahrenheit temperatures $22°$; $35°$; $-17°$; $-32°$. Give the Fahrenheit temperatures corresponding to Centigrade temperature $-25°$; $-12°$; $-5°$; $20°$; $85°$.

19. What temperature has the same reading on both scales?

The teacher should also have each student take some problem in his own experience: the number of inhabitants in his town; the number of pupils in his school; the earnings in his father's store; the price of wheat or of cattle; the amount of rainfall in various years; or any other familiar instance of varying quantity. This problem should be carefully investigated by the student, and the figure should be drawn.

The World Almanac, which can be had by addressing *The World*, New York City, contains much valuable information of the kind needed in such problems.

18. Equation and Graph. In the case of prices of a commodity, such as in the example in § 17, we may also represent the cost of any number of pounds by an equation. Let c be the cost in cents, p the number of pounds; then if butter costs 30¢ per pound,

$$c = 30\,p.$$

This equation represents the same thing (cost of an amount of butter) as Fig. 10; hence, we say that this equation *belongs to* that figure, or, briefly, *this is the equation of that figure;* and that figure is *the graph of this equation.*

Notice that the figure may easily be drawn from the equation. Taking various simple values of p, we get the following table:

Pounds	0	1	2	3	10	etc.	-1	-2	etc.
Cost	0	30	60	90	300	etc.	-30	-60	etc.

If we plot these points, as before, we get a figure like Fig. 12.

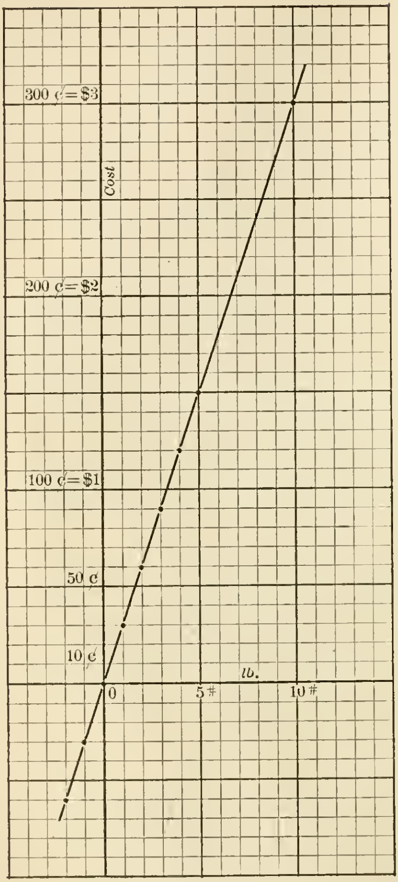


Fig. 12.

It will be evident to the student from the figure that this graph is a straight line through the starting point, as mentioned in § 17. The proof of this fact is given in § 80, p. 140.

19. Equations of Prices. We proceed to extend the suggestion of § 17 that *most price curves are straight lines.* If the cost of one article (or of one unit of measure of a measurable commodity) is known, the cost of any number is given by the following equation, where c is the total cost, k the cost of one, and n the number bought:

total cost = (known cost of one) × (number),

or, $$c = k \cdot n.$$

As in the example of § 18, the graph of such an equation is always a straight line through the starting point. (See also § 80.)

20. Linear Equations. Straight Lines. The student should notice that *ordinary proportion** might be used in the preceding examples. Other examples in which proportion might be used lead to equations similar to those above, and to straight line graphs.

Ex. 1. If a man walks 3 miles per hour, and if d denotes the total distance (in miles), and n the total number of hours he walks, evidently $d = 3n$.

Ex. 2. If x denotes the number of feet in a certain length, and y denotes the number of inches in the same length, $y = 12x$.

[Let the student make a table of values and draw a graph for each of these examples.]

Many examples arise in business and in science in which ordinary proportion could not be used directly. Such examples may lead to straight line graphs.

Ex. 3. If the cost of setting the type for a circular is \$2.00 and the cost of paper and printing is $\frac{1}{2}$¢ per copy, find the cost of any number of copies.

Let c mean the total cost in cents; let n be the number of copies. Then,

$$c = \tfrac{1}{2}n + 200.$$

In order to plot the graph of the equation $c = \frac{1}{2}n + 200$, let us call c_1 the cost of printing and paper alone, then $c_1 = \frac{1}{2}n$ and the figure is a straight line, as above. Now the effect of the cost of setting the type is to increase the price by 200¢, no matter what the number of copies. Hence, the figure for the total cost is found by simply *raising* the whole figure by an amount that denotes 200¢ on the vertical scale. Hence, the final figure is also a straight line.

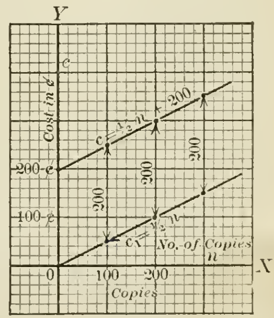


FIG. 13.

* It is assumed that the student is familiar with ordinary proportion as treated in all arithmetics. If not, the teacher may well recall it to his mind.

All of the equations of §§ 18, 19, 20 are of the form

$$y = kx + l$$

where k and l are fixed numbers. The graph for y and x is a straight line. For this reason any equation of the form $y = kx + l$, or any equation that can be reduced to this form, is called a **linear** equation or a **simple** equation.

EXERCISES III: CHAPTER II

1. If 11 tickets to an entertainment cost \$2.75, how much do 5 such tickets cost?

Let n be the number of tickets bought, and c the total cost (in cents); then $c = kn$, where k is the price of 1 ticket (in cents). On a figure, as before, plot the values $n = 11$, $c = 275$ (cents), given in the problem, at P. Join P and O; the line PO represents the cost of any number of tickets. Go 5 units to the right of O and then up to the point Q; opposite Q is the point which represents the number 125 on the vertical line. Hence, 5 tickets cost 125 cents.

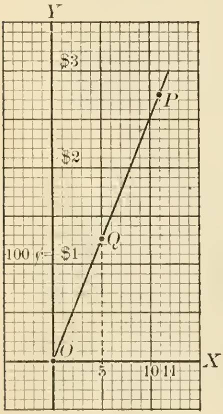


Fig. 14.

We might have found this from the equation. For $c = kn$, and $c = 275$ when $n = 11$; hence, $275 = 11\,k$,

or, $k = 25$.

If now $n = 5$, we have

$$c = 25\,n = 25 \cdot 5 = 125,$$

which is the result just found.

The advantage of the graph is that it solves all such problems (approximately) at once. Again, the figure quickly shows the answer to many reverse problems. Thus, if a man has \$2.40, how many tickets can he buy? Look for the point for 240 on the vertical line, pass over to the line OP, then look down at the horizontal. It is clear that he can buy only 9, for the value found is less than 10.

2. Draw a graph to show the relation between kilograms and avoirdupois pounds. What is the equation expressing the former in terms of the latter? Compare Ex. 2, p. 25.

[SUGGESTION. Let x denote the number of kilograms in a given weight, and let y denote the number of pounds in the same weight. Then $y = 2.2 \times x$ (nearly), since 1 Kg. = 2.2 lb. See Tables.]

3. Express approximately in kilograms 3 lb., 7 lb., 11 lb. Express approximately in pounds 0.5 Kg., 3 Kg., 4.5 Kg.

4. What is the equation representing the area A of a rectangle whose base is 5 and altitude a? Draw the graph.

5. Find A in Ex. 4 if $a = 2$; if $a = 5$; if $a = 3\frac{1}{2}$. Find a if $A = 15$; if $A = 27\frac{1}{2}$.

6. Express by an equation the relation between United States dollars and English pounds sterling. Compare with Ex. 2. Draw the graph.

7. Express by an equation the relation between the difference in time of two places on the earth's surface and the difference of longitude. Draw the graph. Interpret the meaning of positive and negative values of each difference.

[SUGGESTION. A difference of 15° in longitude makes a difference of 1 hour in time.]

8. Expressing longitude east of Washington, D. C., as *positive*, and longitude west as *negative*, find the time corresponding to noon at Washington at a place whose longitude is $+30°$; $-30°$; $+25°$; $-40°$. Find the longitude of a place whose time differs from that at Washington by $+3$ hr.; -3 hr.; $+2\frac{1}{4}$ hr.; $-5\frac{1}{3}$ hr.

9. Represent by a graph the distance traversed by a bicyclist at the rate of 8 miles an hour. What is the equation connecting the distance and the time?

10. In what time will the bicyclist cover 2.3 miles? 3.2 miles? How far will he go in $1\frac{1}{2}$ hours? in $3\frac{1}{4}$ hours?

11. The cost of setting type for an order of business cards is \$0.75; the cost of cards and printing is $\frac{1}{4}$¢ apiece. What is the equation for the cost of any number of printed cards? Draw the graph. Estimate the price of 150 cards; of 500 cards; of 650 cards.

12. Another firm offers to disregard the initial cost of typesetting and to print the cards for $\frac{1}{2}$¢ apiece. What is the cost of any number of copies? Draw the graph on the same diagram as that for Ex. 9.

13. Which firm will most cheaply print 100 cards? 500 cards? Which firm will do the most printing for $1.25? for $2.00? How many cards will be printed just as cheaply by one firm as by the other, and what will be the cost of the printing?

14. The initiation fee in a certain club is $3.00; the yearly dues are $2.00. What is the cost of membership in the organization for any number of years? Draw the graph.

15. Find the cost of membership in Ex. 14 for 5 years; for 12 years; for 15 years. Find the length of membership of a member who has paid $11; $29.

16. The bicyclist of Ex. 9 has a distance of 20 miles to go. Represent by an equation the relation between the time he rides and the distance that *remains* for him to travel. Draw the graph.

17. How far has he left to go after traveling .82 hour? 1.77 hours? How long must he travel in order to be within 12 miles of his destination? 4 miles?

18. Draw in the graph whatever should correspond to the bicyclist's riding past his destination. How long after starting will he have left -4 miles to travel to his destination? -12 miles?

19. Two trains are running at the same rate in the same direction. If x is the distance passed over by one train in any length of time, and y the distance passed over by the other in the same time, what is the relation between y and x? Draw the graph showing this relation.

20. Two trains start from Chicago and run at the same rate in *opposite* directions. If x and y represent the distances of the two trains from Chicago at any time, express the relation between y and x by an equation and by a graph.

21. Suppose that one train starts from Boston when another is 5 miles ahead, and they travel at the same rate in the same direction. When the first train is x miles from Boston, how far is the other from Boston? If y is the distance of the second train from Boston, what is the relation between y and x? Draw the graph.

22. A problem similar to the preceding leads to the equation $y = x + 9$. Plot the graph. When $x = 3$, what is y? When $y = 17$, what is x?

23. From a time-table of some railroad between two important cities, construct a figure to show the movements of trains; mark the distances, starting from one city along the main horizontal line, and mark the times of day from 12 o'clock of one day to 12 o'clock the next day on the main vertical line. Then plot the position of each train at the time shown in the time-table.

Plot graphs corresponding to the following equations (plot only positive values unless you know how to perform the necessary operations with negative values):

24. $y = x - 2$. When $y = 7$, what is x? When $y = 5$, what is x?

25. $y = 4x - 2$. Plot only points for which x is positive. What is y when $x = 2\frac{1}{2}$? when $x = 3\frac{1}{4}$?

26. $2y = x + 3$. (This may also be written, $y = \frac{1}{2}x + \frac{3}{2}$.) What is y when $x = 1$? 3? 5? What is x when $y = 2$? 3? 4?

27. $5y = 2x + 10$. What is y when $x = 0$? 2? 5? What is x when $y = 4$? 6?

21. Other Examples. Some examples do not lead to straight line graphs. Those which give a linear equation (§ 20) do, of course; but it is easy to make examples which lead to other kinds of equations and to other kinds of figures. The following example will dispel the false idea that all graphs are straight lines.

Ex. 1. What is the area, A, of a square in terms of its side, s? Draw the graph.

Solution. If A is the area (in sq. ft.), and s is the length of one side (in ft.), we have $A = s^2$. Giving s various values, we find:

Length of side (s):	0	1	2	3	4	5	etc.	$\frac{1}{2}$	$\frac{3}{2}$	etc.
Area of square (A):	0	1	4	9				$\frac{1}{4}$		
Point in Fig. 15:	0	A	B	C	D	E				

[Let the student fill in the blank spaces and extend the table.]

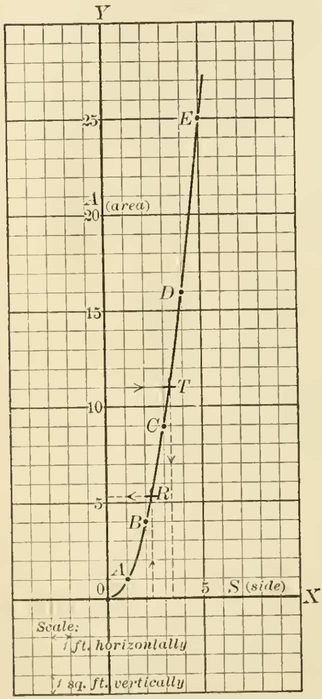


Fig. 15.

Plotting these values of s and A, we obtain a graph that is *not a straight line* (Fig. 15). This figure may be used as above; thus, if the side is 2.3, we may find the area by going out 2.3 on the horizontal line to the right, then up to the curve (R in Fig. 15), then over to the vertical line; the value of A is seen to be a little over 5; really, $A = 5.29$. Again, if $A = 11$, we can find $s = 3.3$ (about); the point is T in Fig. 15.

22. Review Exercises. The following exercises lead to equations like those in §§ 18, 19, 20, that is, the graphs are straight lines. When one of the quantities that vary is given, the other can be found, either from the equation or (*approximately*) from the figure. The two answers should be the same, except for the slight inaccuracy of the figure; thus the figure serves as a check.

REVIEW EXERCISES IV: CHAPTER II

1. Represent by a graph the multiplication table for multiplier 7, from 7×0 to 7×12. Read off 7×4.5; 7×2.5; $7 \times 3\frac{1}{3}$.

Determine a number, n, such that $7n = 85.5$; $7n = 3.5$.

2. A certain grade of cloth costs 20¢ a yard. Represent in a figure the cost of any number of yards. Read from the figure the cost of 3.5 yards; -5.5 yards; 1.4 yards. (Write equations.)

How many yards cost 64¢? 86¢? -50¢?

3. The value of farm property in the United States is given at intervals of ten years from 1850 to 1900 in the following table:

Year	1850	1860	1870	1880	1890	1900
Billions of Dollars	3.98	7.98	8.90	12.18	16.08	20.44

Draw the graph; estimate the values for intermediate years, recording your results in the form of a table.

4. Represent graphically the relation between the circumference of a circle and its radius. Use the figure to determine approximately the radius of a circle whose circumference is 3 ft.; 22 cm.; 35.5 units. (See Table.)

5. Letting d represent one degree, what is the circumference of a circle in terms of d? If r is the radius of the circle, what is the circumference in terms of r? What relation therefore holds between d and r? Draw the picture.

Express as a multiple of the radius of the circle an arc of 57.3°; 100°; 172°. How many degrees are there in an arc twice as long as the radius?

6. Represent by a picture the relation between pounds and ounces. Construct and solve problems in the reduction of pounds to ounces and ounces to pounds. Compare Ex. 2, p. 28.

7. Find out the rate per \$100 valuation of your city, county, and state taxes. Plot a figure to represent the tax on

any valuation by representing the valuation along the horizontal line (\$100 to one large space) and the tax along the vertical line (\$1 to one large space). Draw a line to indicate the total tax. Express the same facts by equations. Estimate the taxes on a house and lot valued at \$2750. Find the value of property taxed at \$6.25.

8. A man rides horseback for an hour and a half at the rate of 8 miles an hour; then, leaving his horse, he walks back at the rate of 4 miles an hour. Draw a picture showing the distance, d, of the man from his starting point at any time, t. How long does the trip take?

9. Another man starts at the same time from the same place as the man in Ex. 8, walking at the rate of three miles an hour in the same direction. How long after the start will the first man, returning, meet the second, and how far from the starting point will the meeting take place? Solve only graphically.

10. A train that leaves Kansas City at 10 A.M. arrives in St. Louis at 6 P.M. The distance is 280 miles. Find the average speed. If t denotes the time (in hours) after leaving Kansas City, and d denotes the distance (in miles) from Kansas City, express by an equation and by a graph the relation between d and t. Find d when $t = 2\frac{1}{2}$. Find t when $d = 105$; when $d = 200$.

11. A train leaves New York at 2 P.M. and arrives in Chicago (940 mi.) at 5 P.M. the next day. Express by an equation and by a graph the relation between distance and time. When should the train reach Buffalo (430 mi. from New York) if its speed is never changed?

12. Plot the graphs of the equations $y = 2x - 9$, $y = x - 3$, on the same diagram.

What pair of values of x and y satisfies both equations, *i.e.* belongs to both figures?

13. Proceed as in Ex. 12 in case of the equations:

$$y = 10 - x, \quad y = x - 7.$$

SUMMARY OF CHAPTER II: MEASUREMENT AND AIDS IN EXPRESSION, pp. 13–32

CHAPTER III. ADDITION AND SUBTRACTION; SIMPLE EQUATIONS

PART I. GENERAL RULES FOR OPERATION; PARENTHESES

23. Extension of the Operations. The student has seen (§ 15, p. 15) how to add and subtract in several new cases.

Fig. 16.

We have seen that we go *forward* to *add* a positive number, *backward* to *subtract* a positive number; thus, $2+3=5$, $-1+3=2$, $2-5=-3$, etc. Let us carry out these processes systematically, proceeding to perform addition, subtraction, etc., in cases not mentioned in elementary arithmetic. *In doing so we shall think of* **addition**, *for example, as the same operation as that used in arithmetic, extended now to all the numbers we know, the extension being carefully made in order that the most essential properties of elementary addition remain true; these properties are mentioned below.*

24. Properties. One important characteristic of addition is the fact that the numbers added may be taken in any order without changing the result.

Thus, $3+4=4+3$, for both equal 7. The same thing is true in all additions of elementary arithmetic.

Instead of writing down every possible example, we may say that $a+b=b+a$, if a and b mean any two numbers whatever.

This fact is often used to check the correctness of additions, for if a problem is solved by adding the numbers in two different orders, the results should be the same. Thus, one often adds a column from the bottom upward, and then also from the top downward.

The most fundamental properties, including, for convenience, the properties of multiplication, are the following:

I. $a + b = b + a$.

This is mentioned just above. Example: $3 + 4 = 4 + 3$.

II. $a \times b = b \times a$.

This has to do with multiplication. Example: $5 \times 7 = 7 \times 5$.

III. $a + (b + c) = (a + b) + c$.

Example: $2 + (7 + 4) = (2 + 7) + 4$, that is, $2 + (11) = (9) + 4$.

This rule, also, is often used in adding columns of figures; thus, instead of adding the numbers one by one, we may add groups, if convenient; for example, if a figure 7 follows a figure 3 we add 10, instead of adding 7 and then 3.

IV. $a \times (b \times c) = (a \times b) \times c$.

Example: $3 \times (5 \times 4) = (3 \times 5) \times 4$, that is, $3 \times 20 = 15 \times 4$.

This is often used; thus, $(5 \times 8) \times 6$ is easier than $5 \times (8 \times 6)$.

V. $a(b + c) = ab + ac$.

Example: $5(3 + 7) = 5 \times 3 + 5 \times 7$, that is, $5(10) = 15 + 35$.

This is often used; thus, $52 \times 7 = 50 \cdot 7 + 2 \cdot 7$, which is easier. This is really the principle of multiplication taught in arithmetic.

These rules are called **axioms**, because they are merely assumed to be true; there is no pretense of proving them. We take these rules in preference to any others because experience has shown that they are the simplest and most useful. See also the footnote on p. 56.

These rules are frequently named as follows: I. *Commutative Law of Addition*; II. *Commutative Law of Multiplication*; III. *Associative Law of Addition*; IV. *Associative Law of Multiplication*; V. *Distributive Law.* We shall not often use these names.

EXERCISES I: CHAPTER III

Show that the above rules hold if:

1. $a=7,\ b=3,\ c=5.$

2. $a=0,\ b=2,\ c=7.$

3. $a=\frac{3}{4},\ b=\frac{7}{5},\ c=\frac{2}{3}.$

4. $a=1,\ b=2,\ c=3.$

By means of Rule V:

5. Multiply 28 by 5.

6. Multiply 63 by 8.

7. Multiply 75 by $\frac{1}{5}$.

8. Multiply $6\frac{2}{3}$ by 7.

Use the above rules in order to perform the operations indicated:

9. $3\times(4\times 3\frac{1}{3})$.

Solution. $3\times(4\times 3\frac{1}{3})=3\times(3\frac{1}{3}\times 4)=(3\times 3\frac{1}{3})\times 4=10\times 4=40.$

10. $12\frac{1}{2}\%$ of \$200.

Solution.

$12\frac{1}{2}\%$ of $\$200=(12\frac{1}{2}\times\frac{1}{100})\times\$200=12\frac{1}{2}\times(\frac{1}{100}\times\$200)=\$25.$

11. $2+(8+7)$.

12. $5\times 7+5\times 3$.

13. $16\frac{2}{3}\%$ of \$300.

14. $[8+(12+4)]\times 5$.

15. $4\frac{1}{2}\times 4\frac{1}{2}$.

16. 53×53.

25. Addition and Subtraction: Negatives. As mentioned in § 23:

(1) *To add a positive number we go forwards on the scale.*

(2) *To subtract a positive number we go backwards on the scale.*

We now add to these the following:

(3) *To add a negative number we go backwards on the scale*, by the amount indicated by the number.

Thus, 9 dollars + 3 dollars debt = 9 dollars + (− 3 dollars) = 6 dollars, which is the same as 9 dollars − 3 dollars. Likewise, $9\,d+(-3\,d)=6\,d=9\,d-3\,d$, no matter what d means. This scheme is useful. The student will see also that in this process the rules mentioned above hold good. For example, $(9\,d)+(-3\,d)=(-3\,d)+(9\,d)$ by I; this is true, for each sum is $6\,d$.

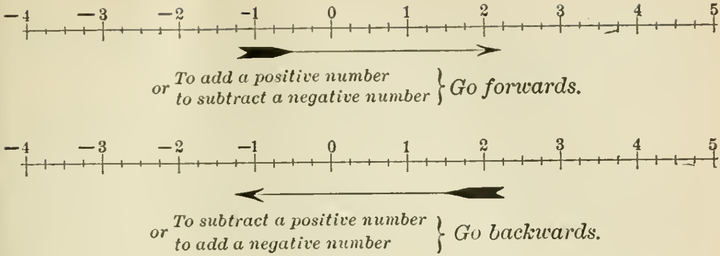


FIG. 17.

(4) *To subtract a negative number we go forwards on the scale*, by the amount indicated by the number.

Thus, *subtracting* (or removing) a debt amounts to *increasing* the wealth of the man who owed it. If you have ten dollars and owe four dollars, your total wealth is six dollars. But if some one pays the debt for you, you will really have ten dollars; *i.e.* $6\,d - (-4\,d) = 10\,d$.

(5) *To subtract any number, change its sign and then add the resulting number.* This rule is very important, as will be seen later.

26. Addition and Subtraction of Several Numbers. In business and ordinary life we often need to add several numbers some of which are positive and some negative. Thus, if a merchant has several debts and also several different amounts of money and valuable goods, all the debts are added together to find the total debts, and the values of all the money and goods are added together to find the total "assets." The difference between these amounts represents the real wealth: it is a *debt* (*i.e.* negative wealth) if the total debts are greater than the total assets; it is real wealth (*i.e.* positive), if the total assets are the greater. Some examples are given below in which this process proves useful.

When several numbers are to be added together, we add all the positive numbers to make a positive total, then all the

negative numbers to make a negative total; the sum sought is the difference of the amounts of these totals with the sign of the greater one prefixed to it.

Thus, the sum of $6\,d$, $-7\,d$, $4\,d$, $-10\,d$, $25\,d$, is found as follows:

$6\,d + 4\,d + 25\,d = 35\,d$, the total of positive numbers.

$-7\,d - 10\,d = -17\,d$, the total of negative numbers.

$35\,d + (-17\,d) = 35\,d - 17\,d = 18\,d$, the final answer.

Again,

$3\,d + (-5\,d) + 8\,d + (-12\,d)$, is found as follows:

$3\,d + 8\,d = 11\,d$, total +

$-5\,d - 12\,d = -17\,d$, total −

$11\,d + (-17\,d) = -(17 - 11)\,d = -6\,d$, the final answer.

EXERCISES II: CHAPTER III

Perform the operations indicated. In each case draw a figure showing the scale.

1. $10 - (-8)$.

2. $3 - 4$.

3. $4 + (-3)$.

4. $2 - (-2)$.

5. $-3 - (-4)$.

6. $-3 + (-4)$.

7. $8 - (-7) + (-5)$.

8. $7 - 0$.

9. $42 - (-36)$.

10. $\$215 - \$472$.

11. $-65\,d + 39\,d - (-12\,d)$.

12. $17\,x - (-35\,x)$.

13. $12\,b + (-7\,b) + (-5\,b)$.

14. $6\,a - (-0\,a)$.

15. $0\,a - (6\,a)$.

16. $0\,a - (-6\,a)$.

17. What will be the total wealth in money of a man who has \$1500 in the bank and owes \$1200, if by a transfer of property he cancels \$750 of a debt?

18. A weight of 30 lb. is attached to three small balloons which pull up respectively 10 lb., 15 lb., and 20 lb. What is the upward pull of the whole device? Hence, what may we say is the weight (downward force) of the whole device? Express the problem in terms of positive and negative numbers, using downward force as *positive*.

19. A vessel capable of making 10 miles an hour is opposed by a current of 6 miles an hour. By hoisting sails, use is made of the wind blowing in the same direction at 8 miles an hour. What is the speed at which the vessel moves? Express in terms of positive and negative numbers.

20. A vessel is making use of both sails and steam; the latter alone will propel it at 15 miles an hour, the wind opposes the motion at the rate of 10 miles an hour, and an opposing current is flowing at 2 miles an hour. What is the speed of the vessel? The sails are lowered. What now is the speed? Express in terms of positive and negative numbers.

27. Addition and Subtraction of Negative Numbers. By the rules above:

(1) *Adding a negative number is equivalent to subtracting a positive number of equal amount; they both mean going backward on the scale by the same amount.*

(2) *Subtracting a negative number is equivalent to adding a positive number of equal amount; they both mean going forward on the scale.*

Thus,
$$5 + (-2) = 5 - 2 = 3,$$
$$5 - (-2) = 5 + 2 = 7,$$
and so on.

This holds equally well for dollars (d is used below) or for any other unit quantity:

$$5\,d + (-2\,d) = 5\,d - 2\,d = 3\,d.$$
$$5\,d - (-2\,d) = 5\,d + 2\,d = 7\,d.$$

It follows that additions and subtractions of negative numbers may be turned into subtractions and additions of positive numbers by changing the sign of the given quantity.

Hence, we never really distinguish between such expressions as

$5 + (-2)$ *and* $5 - 2$; *or, in general,* $a + (-b) = a - b$;

nor between

$$5-(-2) \text{ and } 5+2; \text{ or, in general, } a-(-b)=a+b.$$

We also remark that

$$5-(+2) \text{ means } 5-2; \text{ or, in general, } a-(+b)=a-b.$$

With this understanding, subtractions or additions of either positive numbers or negative numbers may be performed as above, by first thinking of all subtractions as turned into additions.

EXERCISES III : CHAPTER III

1. A man has \$215 in one bank, \$134 in another, and goods worth \$1250; there stand against him a debt of \$75, and a mortgage of \$750. Using d for one dollar, find his total wealth.

2. Letting d stand for one dollar, complete the following table, which represents increases (+) and diminutions (−) of the wealth of a family of three brothers:

	JAN.	FEB.	MAR.	TOTAL FOR THREE MONTHS
A. . .	$912\,d$	$-14\,d$	$36\,d$	
B. . .	$-721\,d$	$255\,d$	$-1000\,d$	
C. . .	$2200\,d$	$-133\,d$	$756\,d$	
TOTAL .				

3. A weight of 20 g. (grams) in a certain piece of mechanism is to be raised by two wires pulling upward by 6 g. and 8 g. respectively, and a spring exerting an upward pull of 10 g. With what upward force will the weight rise if the smaller wire has been broken? Will the total force be exerted up or down, and what will be its magnitude?

4. If p denotes one pound, the weight of the basket of a balloon is $320\,p$, of the instruments $68\,p$, of each of 9 sand

bags $100\,p$; the weight of the balloon itself is $-1500\,p$. What is the total weight of balloon and contents?

5. Let m denote miles an hour. A boat using both sails and steam is urged forward by its steam power at the rate of $12\,m$, and by the wind at the rate of $4\,m$. It sails against a current flowing at the rate of $5\,m$. What is its rate of progress?

Perform the following additions:

6. $12 + 9 + (-7) + (-13) + (-2)$.

7. $(-7) + (-9) + 5 + (-2) + 20$.

8. $8\,d + 7\,d + (-20\,d) + 6\,d + (-3\,d)$.

9. $(-7\,x) + 4\,x + 10\,x + (-5\,x) + 12\,x$.

10.

(a)	(b)	(c)	(d)
15	-5	$7\,b$	$2\,z$
-18	-4	$-4\,b$	$-8\,z$
32	-3	$8\,b$	$5\,z$
-17	-2	$12\,b$	$9\,z$
-5	10	$-3\,b$	$-17\,z$

(e)	(f)	(g)	(h)
6	4.3	$16\,p$	$-12\,f$
-7	-7.25	$-18\,p$	$-10\,f$
-9	-6.8	$12\,p$	$75\,f$
-15	10.53	$3\,p$	$-68\,f$
20	-8.2	$-15\,p$	$-\,f$

11. The *average* of two numbers is their sum divided by 2. Find the average of 4 and 6; of -4 and 6; of 10 and 15; of -10 and 15; of -4 and -6; of -10 and -2; of $5\,d$ and $7\,d$; of $7\,d$ and $-5\,d$; of $-7\,d$ and $-5\,d$.

12. The average of several numbers is their sum divided by the number of numbers added. Find the average of 2, 3, 4, and 5; of -2, 3, -4, 5; of $5\,d$, $-7\,d$, and $3\,d$; of $-10\,d$, $-12\,d$, and $-4\,d$.

13. In Ex. 2, find the average gain per month of A during the three months given; of B; of C; of the whole family.

14. In Ex. 2, find the average gain of the three men during January; during February; during March. Find the average gain for the three months.

15. Find the average of the numbers added in Ex. 6; in Ex. 7; Ex. 8; Ex. 9; in each part of Ex. 10.

16. In Ex. 2, how much above the average earnings during January were A's earnings during that month? How much were C's above the average? How much were B's *below* the average? Can you state that B's earnings were a certain amount *above* the average by using a *negative* number?

17. A business house gains $5000 one year, $2000 the next; it loses $1500 the third year, and $3000 the fourth; the fifth year it gains $1000. What is the total gain for the five years? Find the average yearly gain for the five years.

18. In Ex. 17, how much above the average gain per year was the gain during the first year? during the second? third? fourth? fifth?

19. A business house during three successive years loses $500, $300, and $350. What is the total loss? the average loss? What is the total gain, if the results are expressed as gain? the average gain? State the result as a problem in averaging negative numbers.

20. The results of a business for five years are as follows:

	FIRST YEAR	SECOND YEAR	THIRD YEAR	FOURTH YEAR	FIFTH YEAR
Gain	$1732				$3000
Loss		$5251	$2572	$135	

What is the total net *gain*? the average gain? State the result as a problem in averaging positive and negative numbers.

21. Find the average of the following temperatures:

$$-3.5°, -2.5°, -1°, -2°, -2.5°.$$

22. What is the average temperature in the example of § 16, p. 17? in the example on p. 18?

23. A rifle gives the ball the muzzle velocity of 1200 feet per second. Fired directly backwards from a moving car, the actual velocity of the ball is 1156 feet per second; how fast is the car moving?

24. Fill in the spaces left *blank* in the following account of the gains of a merchant at his three stores:

	First Store	Second Store	Third Store	Total
1902	$5,000	$10,500	$	$16,000
1903	$3,000	$ 8,000	$	$11,525
1904	– $ 50	$ 6,250	$	$
1905	– $2,250	– $ 1,050	$	– $ 4,500
1906	$	$	$	$ 1,000
Total gain of five years	$8,700		$	$28,225
Average gain per year		$4,540		

25. The total weight of a balloon and basket containing a man, apparatus, sand bags, etc., and small balloons, is $-25{,}000\,p$, where "p" denotes one pound. The man weighs $150\,p$, the basket and apparatus $50\,p$, the ten sand bags each weigh $75\,p$: and the ten balloons each $-5\,p$. What is the weight of the large balloon?

26. Carry out the following subtractions:

(*a*) $\begin{array}{r}15\\-30\\\hline\end{array}$	(*c*) $\begin{array}{r}-25\\-17\\\hline\end{array}$	(*e*) $\begin{array}{r}17\\-25\\\hline\end{array}$	(*g*) $\begin{array}{r}17\\25\\\hline\end{array}$	(*i*) $\begin{array}{r}5x\\-\ 6x\\\hline\end{array}$
(*b*) $\begin{array}{r}-25\\17\\\hline\end{array}$	(*d*) $\begin{array}{r}25\\-17\\\hline\end{array}$	(*f*) $\begin{array}{r}-17\\-25\\\hline\end{array}$	(*h*) $\begin{array}{r}-17\\25\\\hline\end{array}$	(*j*) $\begin{array}{r}-15p\\-25p\\\hline\end{array}$

27. Reduce the following expressions to simpler forms:

(a) $7x+(-11x)-(-12x)+(-5x)-6x$.

(b) $-25b-(-30b)+12b-16b-(-b)$.

(c) $10d+7d-(-5d)+(-30d)$.

(d) $(-5)+(-5)+(-5)+(-5)+(-5)$.

(e) $(-3k)+(-3k)+(-3k)+(-3k)$.

(f) $125z-(-375z)+(-752z)-502z$.

(g) $13r-50r-(-65r)+73r-11r-(-12r)$.

[Check each exercise by substituting some number for the letter that occurs both in the exercise as given and in your answer.]

28. Addition of Monomials. We have done one thing above which should be stated carefully. Having given $9d+3d$, we write as answer $12d$; and this is correct whether d means dollars, dimes, or anything else. Thus,

$$9d+3d=(9+3)d=12d,$$

$$14x+-11x=[14+(-11)]x=3x,$$

and so on.

Rule. *To add terms that have a common factor, add the coefficients of that factor; the sum is the sum of the coefficients times the common factor.*

This is, in fact, merely another statement of V, p. 35; *e.g.*

$$9d+3d=(9+3)d=12d.$$

The same rule applies for the sum of any number of terms.

Thus, $(6d)+(-7d)+(4d)+(-10d)+(25d)=[6+(-7)+4+(-10)+25]d=18d$, as on pp. 37–38. Or again, $7x+(-12x)+6x+4x=[7+(-12)+6+4]x=5x$; etc.

This rule applies no matter how complicated the units may be; thus, if we are talking of special objects it remains true that *six* of them plus *three* of them $=$ *nine*

of them. Or again, if we are speaking of expressions, however complicated, such as b^2x^3, it remains true that $6\,b^2x^3 + 3\,b^2x^3 = 9\,b^2x^3$, etc.

We use this rule for the addition of *similar* terms. Let the student read again and state the definition of similar terms, § 8. By proper choice of a common factor many terms not similar as a whole may be made similar in part of the letters. Thus, by V, p. 35, $ax + bx = (a + b)x$; $abx^2y + cdx^2y = (ab + cd)x^2y$; etc. The coefficient is chosen such as to make the terms similar.

A simple check on the answer is obtained by putting in various numbers at random in the place of the letters. Thus, if $b = 3$ and $x = 2$ in the preceding example, then $b^2 = 9$, $x^3 = 8$, $b^2x^3 = 72$; and $6\,b^2x^3 = 432$, $3\,b^2x^3 = 216$, $9\,b^2x^3 = 648$. Now $432 + 216 = 648$; if, instead, we had found an untrue addition here, we should know we had made an error in the work above.

Such a check is not an absolute proof of correctness, however; thus, $2\,x + 3\,x = 4\,x + 1$ is not correct in general, although we find it is correct for one choice of x, namely, for $x = 1$.

29. Subtraction of Monomials. Since subtracting a quantity is equivalent to adding its negative, we may subtract by turning all subtractions into additions of the opposite kind of quantity.

Thus, 6 dollars + (– 3 dollars) = 6 dollars – (+ 3 dollars), and 6 dollars + (+ 3 dollars) = 6 dollars – (– 3 dollars), as has been noticed above; or, in general,

$$\begin{cases} a + (-b) = a - b, \\ a + (+b) = a - (-b). \end{cases}$$

The rule given above holds for subtraction also, it being understood that to subtract a quantity means the same as to add its negative. (See § 25, p. 37.)

EXERCISES IV: CHAPTER III

Perform the following additions:

1. $$\begin{array}{r} 6\,d \\ -7\,d \\ 18\,d \\ -5\,d \\ \hline \end{array}$$

2. $$\begin{array}{r} -9\,k \\ -3\,k \\ 13\,k \\ -k \\ \hline \end{array}$$

3. $$\begin{array}{r} 4\,k^2x \\ 7\,k^2x \\ -13\,k^2x \\ 3\,k^2x \\ \hline \end{array}$$

4. $$\begin{array}{r} 3\,abz \\ -10\,abz \\ 12\,abz \\ -15\,abz \\ \hline \end{array}$$

5. $$\begin{array}{r} -3.5\,x^2y^3 \\ -1.7\,x^2y^3 \\ 2.9\,x^2y^3 \\ \hline \end{array}$$

6. $$\begin{array}{r} 2\frac{1}{2}\,gt^2 \\ -3\frac{1}{3}\,gt^2 \\ 2\frac{5}{6}\,gt^2 \\ \hline \end{array}$$

7. $$\begin{array}{r} 1.35\,\dfrac{pv}{T} \\ -2.7\,\dfrac{pv}{T} \\ \hline \end{array}$$

8. $$\begin{array}{r} -1.42\,ab^2c \\ 2.14\,ab^2c \\ -1.98\,ab^2c \\ \hline \end{array}$$

9. $$\begin{array}{r} 0.16\,m^3(n+r) \\ -3.24\,m^3(n+r) \\ -2\frac{7}{8}\,m^3(n+r) \\ \hline \end{array}$$

10. $$\begin{array}{r} as^2t^3 \\ -bs^2t^3 \\ cs^2t^3 \\ \hline \end{array}$$

11. $$\begin{array}{r} 6\,(ax^2+by^2) \\ -8\,(ax^2+by^2) \\ 3\,(ax^2+by^2) \\ \hline \end{array}$$

12. $$\begin{array}{r} -3\,(fx+gy+hz) \\ -8\,(fx+gy+hz) \\ 7\,(fx+gy+hz) \\ \hline \end{array}$$

13. $$\begin{array}{r} 4\,(-ap+bq) \\ -5\,(-ap+bq) \\ -(-ap+bq) \\ \hline \end{array}$$

Perform the following subtractions:

14. $$\begin{array}{r} 12\,bcyz \\ -10\,bcyz \\ \hline \end{array}$$

15. $$\begin{array}{r} -7\,x^3yz^2 \\ 3\,x^3yz^2 \\ \hline \end{array}$$

16. $$\begin{array}{r} 3.81\,r^2 \\ -4.21\,r^2 \\ \hline \end{array}$$

17. $$\begin{array}{r} \frac{1}{6}\,m\,(m-1) \\ \frac{1}{2}\,m\,(m-1) \\ \hline \end{array}$$

18. $$\begin{array}{r} -5\,\dfrac{g}{R^2} \\ -11\,\dfrac{g}{R^2} \\ \hline \end{array}$$

19. $$\begin{array}{r} -1.7\,\dfrac{y^2}{b^2} \\ -5.6\,\dfrac{y^2}{b^2} \\ \hline \end{array}$$

20. $$\begin{array}{r} -10\left(\dfrac{1}{r}-\dfrac{1}{R}\right) \\ -12\left(\dfrac{1}{r}-\dfrac{1}{R}\right) \\ \hline \end{array}$$

21. $$\begin{array}{r} 4\,(x+y-2) \\ -3(x+y-2) \\ \hline \end{array}$$

22. $$\begin{array}{r} 6\,y\sqrt{m^2+y^2} \\ -4\,y\sqrt{m^2+y^2} \\ \hline \end{array}$$

23. $$\begin{array}{r} -1.3\,\sqrt{1-y^2} \\ -1.5\,\sqrt{1-y^2} \\ \hline \end{array}$$

24. Subtract the last quantity in each of the first 13 exercises from the sum of all that precede it in the same exercise.

25. Average the quantities, $5\,abx$, $-7\,abx$, $10\,abx$, $-12\,abx$.

26. Average the quantities in each of the first 13 exercises.

27. Average the quantities, $7\,(b^2x^2+a^2y^2)$, $-10\,(b^2x^2+a^2y^2)$, $b^2x^2+a^2y^2$, and $6\,(b^2x^2+a^2y^2)$.

28. Subtract the first quantity in each of the first 13 exercises from the average of the rest of the quantities in the same exercise.

30. Grouping in Addition. If longer expressions are to be added together, we use the fact that *the order in which terms are added is immaterial*, stated in Axioms I and III, p. 35:

I $$a+b=b+a;$$

III $$a+(b+c)=(a+b)+c.$$

Thus, we may inclose any number of terms in parentheses preceded by the sign + (plus); *likewise we may remove parentheses preceded by the sign* +. This means we may add the terms inside a pair of parentheses together before we add their sum to the others, or we may group the terms in any other way we please.

This is done frequently in everyday life. Thus, if a merchant owns three stores, as in the example on p. 48, he may count his liabilities and his assets for all these stores, or he may count them separately for each store. His total wealth will be the result of adding all these, and will be the same by either method of working. This fact is often used to check the work.

Ex. 1. To calculate $44+36+17+33+57$ we may add all together or we may group in parentheses, thus:

$$(44+36)+(17+33)+57=80+50+57=187,$$

or, $$(44+36)+17+(33+57)=80+17+90=187,$$

or, $$(44+33)+36+(17+57)=77+36+74=187.$$

Which of these is the easiest way?

Ex. 2. A merchant has three stores. These have the following amounts; the debts due and mortgages to be counted as negative:

	Cash	Fixtures	Stock	Acc'ts Due	Debts Due	Mortgage	Total
1st Store	\$ 327	\$ 250	\$ 1200	\$ 463	\$ 651	\$ 500	
2d Store	\$ 56	\$ 515	\$ 2100	\$ 590	\$ 980	\$ 2000	
3d Store	\$ 125	\$ 357	\$ 1570	\$ 25	\$ 415	\$ 1400	
Total							

Find (*a*) the total value of each store; (*b*) the sum of these three to get the merchant's wealth; (*c*) the total cash in the three stores and the total of each of the other columns; (*d*) the sum of these totals to get the merchant's wealth. Do your total answers agree?

EXERCISES V: CHAPTER III

Note. The student should check all answers in which letters are used, by substituting numbers for the letters at random, as suggested above.

1. The gains of a merchant's two stores are as follows for a period of five years. Fill in the spaces left blank:

	First Store	Second Store	Total	Average
1902	\$ 1525	– \$ 7530		
1903	\$ 1035	– \$ 5000		
1904	\$ 105	\$ 565		
1905	– \$ 355	\$ 3325		
1906	\$ 870	\$ 9565		
Total for five years				
Average per year				

Perform mentally the following additions, abbreviating by grouping terms as often as possible:

2. $6 + 8 + 4 + 3 + 2$.

3. $12 + 17 + 28 + 35 + 30 + 5$.

4. $11\,d + 6\,d + 9\,d + 3\,d + d + 4\,d$.

5. $17\,xy + 8\,xy + 12\,xy + 23\,xy$.

6. $3\,z^3t + 9\,z^3t + 12\,z^3t + z^3t + 7\,z^3t$.

Collect into single terms:

7. $(12\,bc + 9\,bc + 8\,bc) + (7\,bc - 3\,bc + 2\,bc)$.

8. $(-\,r^2v - 8\,r^2v + 13\,r^2v) + 8\,r^2v + (7\,r^2v - 12\,r^2v)$.

9. $$\left(6\sqrt{\frac{1+x}{1-x}} - 7\sqrt{\frac{1+x}{1-x}} + 2\sqrt{\frac{1+x}{1-x}}\right) + \left(\sqrt{\frac{1+x}{1-x}} - 3\sqrt{\frac{1+x}{1-x}}\right) + \left(2\sqrt{\frac{1+x}{1-x}} - 3\sqrt{\frac{1+x}{1-x}} + 5\sqrt{\frac{1+x}{1-x}}\right).$$

10. Average the expressions:

$(7\,pq - 5\,pq + pq)$, $(8\,pq - 3\,pq - 9\,pq)$, $(7\,pq + pq - 3\,pq)$.

11. Average the expressions: $2\,lmn - 5\,lmn - 8\,lmn$, $-9\,lmn + lmn - 2\,lmn$, $6\,lmn + 4\,lmn - lmn$.

31. Addition of Longer Expressions. We may now add longer expressions by grouping together those terms which are "similar" (or "like") (see § 8, p. 8).

Ex. 1. One farmer has 7 horses, 12 cows, and 4 sheep, another farmer has 6 horses, 5 sheep, 6 cows. Find their total possessions.

We say:
(7 horses + 12 cows + 4 sheep) + (6 horses + 5 sheep + 6 cows) = (7 horses + 6 horses) + (12 cows + 6 cows) + (4 sheep + 5 sheep) = 13 horses + 18 cows + 9 sheep, but we do not try to add cows to sheep or to horses.

Likewise $(7\,h + 12\,c + 4\,s) + (6\,h + 5\,s + 6\,c) = (7\,h + 6\,h) + (12\,c + 6\,c) + (4\,s + 5\,s) = 13\,h + 18\,c + 9\,s$, no matter what h, c, and s mean.

Again $(6\,ax + 20\,b^2x + 5\,ab) + (6\,b^2x + 30\,ax + 7\,ab) = (6\,ax + 30\,ax) + (20\,b^2x + 6\,b^2x) + (5\,ab + 7\,ab) = 36\,ax + 26\,b^2x + 12\,ab$, no matter what a, b, and x mean. Check this addition by trying $a = 2$, $b = 3$, $x = 1$. Try some numbers of your own choice. Notice that there is no attempt to add together the final terms $36\,ax$ and $26\,b^2x$ and $12\,ab$, just as there is no attempt to add cows to horses or to sheep in the first example.

RULE. *To add long expressions, write them in columns with the similar terms in the same column; add the columns separately; write the total sum as the sum of these separate results.*

This was done in Ex. 2, p. 48. Likewise to add the expressions $(2\,abc + 3\,a^2x - 7\,ab - 4\,bc + 5\,cx^2)$ and $(6\,ab - 4\,a^2x + 5\,bc - 2\,cx^2 - 7)$ and $(4\,bc - 3\,ab - 3\,cx^2 + 2\,a^2x + 4)$ we write

$$\begin{array}{lrrrrrr}
\textbf{Given} & 2\,abc & +\,3\,a^2x & -\,7\,ab & -\,4\,bc & +\,5\,cx^2 & \\
 & & -\,4\,a^2x & +\,6\,ab & +\,5\,bc & -\,2\,cx^2 & -\,7 \\
 & & 2\,a^2x & -\,3\,ab & +\,4\,bc & -\,3\,cx^2 & +\,4 \\
\hline
\textbf{Sum} & 2\,abc & +\,a^2x & -\,4\,ab & +\,5\,bc & & -\,3
\end{array}$$

EXERCISES VI: CHAPTER III

Add the expressions given in each of the following exercises:

[Check each result by substituting random numbers for the letters.]

1. $5\,a + 7\,b - 3\,c,\ -12\,a + b - 2\,c.$

2. $3\,x - 4\,y,\ 7\,x + 2\,y,\ -2\,x - y.$

3. $3\,fx^2 - 4\,gy^2 + 7\,hz^2,\ fx^2 + gy^2 + hz^2,\ 2\,fx^2 + gy^2 - 3\,hz^2.$

4. $al - 8\,bm,\ 6\,bm - 2\,cn,\ 4\,cn - 5\,al.$

5. $16\,abc - 19\,xyz,\ -25\,abc - 11\,xyz,\ 15\,xyz.$

6. $19\,ap + 13\,bq - 15\,cr,\ -17\,bq + cr - 5\,ap,\ 8\,bq - 11\,cr - 2\,ap.$

7. $7\,bcyz - 9\,cazx - 3\,abxy,\quad -2\,bcyz,\quad -3\,cazx + 9\,abxy,\ 2\,cazx - bcyz.$

8. $6\,x^2 - 3\,x + 7,\ -5\,x^2 + 2\,x - 3,\ 2\,x^2 - 9x - 5.$

9. $x^2 + 2\,hx + h^2,\ x^2 - h^2,\ h^2 - hx - 2\,x^2.$

10. $7+3t-\frac{1}{2}gt^2,\ -5t+3gt^2,\ 6-2\frac{1}{2}gt^2.$

11. $5(a+b)-7(c+d)+4(x+y), \quad -(a+b)-(c+d),$

$3(a+b)+2(c+d)-5(x+y).$

12. $3(7x-5y)-8(2x-3y)+10(x-y),$

$-(7x-5y)+7(2x-3y)-4(x-y),$

$-(7x-5y)+2(2x-3y)-5(x-y).$

[Also collect your result into only two terms.]

13. $2(-x-3y)-5(x+3y),\ -(-x-3y)+6(x+3y).$

[Collect your result into as simple a form as possible.]

14. $(6ax+9by-3cz)+(6ax+9by-3cz)$

$$+(6ax+9by-3cz)+(6ax+9by-3cz).$$

15. Add a_1x+b_1y, a_2x+b_2y, a_3x+b_3y, by collecting the coefficients of x into one term and the coefficients of y into one term.

The use of *letters with subscripts*, as here, is very common; thus, a_1 (read "a, sub-one"), a_2, a_3, b_2, b_3 (read "a, sub-two," "a, sub-three," "b, sub-two," "b, sub-three," etc.), are very often used in the place of separate letters. They should be treated simply as separate letters, used to indicate separate numbers, and no meaning should be attached to the small figures except to distinguish a_2 from a_3, for example.

32. Subtraction of Longer Expressions. Just as before, to subtract, we proceed as in addition after changing the sign of the quantity or quantities to be subtracted. *If there are several quantities to be subtracted, we must be careful to change the sign of each one before proceeding to the addition mentioned.*

Ex. 1. A firm of stockmen have \$2500 cash, 600 sheep, 325 cows, a mortgage (debt) of \$1500, and they owe 25 horses to another firm. What is their total wealth?

The total wealth is

$$2500\,d+600\,s+325\,c-1500\,d-25\,h,$$

where the letters stand for the article having the initial.

Ex. 2. If one partner, wishing to set up an independent business, takes part of the debts as well as part of the assets: \$500 cash, 125 sheep, 150 cows, a mortgage (debt) of \$300, what is left for the others?

We must *subtract* $(500\,d + 125\,s + 150\,c - 300\,d)$ from $(2500\,d + 600\,s + 325\,c - 1500\,d - 25\,h)$. To do this we change the sign of *each* of the quantities to be subtracted; then we add the result to the original amount.

Original,	$2500\,d + 600\,s + 325\,c - 1500\,d - 25\,h$
Add	$-500\,d - 125\,s - 150\,c + 300\,d$
Answer,	$2000\,d + 475\,s + 175\,c - 1200\,d - 25\,h$

Ex. 3. $(4\,a - 7\,bx + 3\,c^2) - (6\,a + 5\,bx - 2\,c^2)$.

Original,	$4\,a - 7\,bx + 3\,c^2$	
	$6\,a + 5\,bx - 2\,c^2$	(Change signs mentally and add.)
Answer,	$-2\,a - 12\,bx + 5\,c^2$	

This holds no matter what a, b, x, c mean. Try it if $a = 4$, $b = 3$, $x = 5$, $c = 1$.

EXERCISES VII: CHAPTER III

[Check each result by substituting random numbers for the letters.]

Subtract the second expression from the first:

1. $a - b,\ 2\,a - 3\,b$. **2.** $3\,l - 5\,n,\ -l - 4\,n$.

3. $6\,xy - 4\,ab,\ -5\,xy + 7\,ab$.

4. $-3\,kx^2 - 4\,by^2 + 7\,mz^2,\ -kx^2 - 5\,by^2 + 2\,mz^2$.

5. $-11\,xy^2 + 2\,yz^2,\ -10\,xy^2 + 3\,yz^2 + zx^2$.

6. $xyz + abc + lmn,\ xyz + 2\,abc + 3\,lmn$.

7. $x + 2\sqrt{xy} + y,\ x - 2\sqrt{xy} + y$.

8. $a^2 - 2\,ab + b^2,\ b^2 - 3\,ab - 4\,a^2$.

9. $gryz - rpzx - pqxy,\ rpzx + pqxy - gryz$.

10. $x^2 - y^2 - 2\,yz - z^2,\ z^2 + x^2 - 2\,yz + y^2$.

11. $2 + 11\,t - 13\,gt^2,\ 1 + 12\,t - 10\,gt^2$.

12. $6(5\,x - 3\,y) - 2(x + y),\ 5(5\,x - 3\,y) - 3(x + y)$. Then collect your result into two terms.

Perform the operations indicated:

13. $(-9\,ax + 7\,by - cz) - (-8\,ax - by + 2\,cz)$.

14. $(2\,x - y) - (x - 2\,y) + (-x - y)$.

15. $(2\,x - y) - (x - 2\,y) - (x + y)$.

16. $(x^2 - 2\,hx + h^2) - (x^2 - 6\,hx + 9\,h^2) - (x^2 - h^2) + (h^2 - hx)$.

17. $(x^2 - 2\,xy + y^2) - (3\,x^2 - 5\,xy + 2\,y^2) + (2\,x^2 - 3\,xy + y^2)$.

18. Subtract $a_1x + a_2y$ from $b_1x + b_2y$ by collecting the coefficients of x into one term, and the coefficients of y into one term.

33. Removal and Insertion of Parentheses. We saw that parentheses preceded by the sign $+$ may be inserted or removed as we wish, without any other change.

But if a pair of parentheses is preceded by the sign $-$, the whole interior is to be *subtracted* from what goes before it. *We must therefore change the sign of each term inside the parentheses when we take them away.*

Thus, $(4\,a - 7\,bx + 3\,c^2) - (6\,a + 5\,bx - 2\,c^2) = 4\,a - 7\,bx + 3\,c^2 - 6\,a - 5\,bx + 2\,c^2 = -2\,a - 12\,bx + 5\,c^2$, which is the same as the result obtained in § 32. Notice that the *first term* inside a pair of parentheses has the sign *plus* if no sign is written; in removing parentheses preceded by the sign $-$, care should be taken not to overlook the first term.

Similarly, *if we insert parentheses preceded by the sign $-$, we must change the sign of each term we inclose.*

Thus, $4\,a - 7\,bx + 3\,c^2 - 6\,a - 5\,bx + 2\,c^2 =$

$$(4\,a - 6\,a) - (7\,bx + 5\,bx) + (3\,c^2 + 2\,c^2) = -2\,a - 12\,bx + 5\,c^2,$$

which is, of course, the same result obtained before. Notice that the middle pair of parentheses is preceded by the sign $-$, and that the signs of the terms inside of it are changed as they are put inside.

To remove or to insert parentheses, change the sign of $\left\{\begin{matrix}\textit{none}\\ \textit{each}\end{matrix}\right\}$ *of the terms if the sign* $\left\{\begin{matrix}+\\ -\end{matrix}\right\}$ *precedes the parentheses.* (Read "none" with $+$; read "each" with $-$.)

If several parentheses are used, one inside the other, remove the one farthest inside first, and the others in their order. The student should read § 11, p. 10, and should state the forms of parentheses used in algebra.

After considerable practice the student should be expert enough to remove any one pair of parentheses without touching the others; he may then find it more convenient not to use the order specified above.

EXERCISES VIII: CHAPTER III

Remove the parentheses in the following exercises, and simplify as much as possible:

1. $7+(-3+2)$.

2. $7-(3-2)$.

3. $-6+[5-(7+3)+12]$.

4. $13-[7-(2-5)]$.

5. $-11b+[8b-(2b+b)-3b]$.

6. $8kz-[7kz-(3kz-5kz)]$.

7. $8kz-[(7kz-3kz)-5kz]$.

8. $8kz-(7kz-3kz-5kz)$.

9. $[8kz-7kz]-[3kz-5kz]$.

10. $[8kz-(7kz-3kz)]-5kz$.

11. $[6mn^2-(8l^3-mn^2+\overline{3n^3-mn^2})-(22mn^2-8l^3)]$.

12. $13xyz-[(5abc-xyz)-(3xyz+\overline{7abc-xyz})]$.

In the following, remove the parentheses, beginning with the outermost:

13. $-6+[5-(7+3)+12]$. Does your result agree with Ex. 3?

14. $13-[7-(2-5)]$. Does your result agree with Ex. 4?

15. $[6mn^2-(8l^3-mn^2+\overline{3n^3-mn^2})-(22mn^2-8l^3)]$. Does your result agree with Ex. 11?

16. $a-[-(-\overline{-b})]$.

In the following examples rewrite the expressions, inclosing the last two terms first in parentheses, preceded by the plus sign, and then in parentheses preceded by the minus sign:

17. $a+b+c$.

18. $a-b-c$.

19. $a+b-c$.

20. $b-a-c$.

21. $a-b+c$.

22. $xy-xz+z^2$.

PART II. APPLICATIONS. LINEAR EQUATIONS

34. Equations. We may apply the knowledge gained to solve many problems similar to those of Chapter II.

In Chapter II, § 20, we had an equation

$$c = \tfrac{1}{2} n + 200,$$

where c means the total cost in cents, and n the number of copies of a certain printed pamphlet. If the cost were \$4.25, or 425 cents, we could find the number of copies.

(1) $$c = \tfrac{1}{2} n + 200,$$

or, since $$c = 425,$$

(2) $$425 = \tfrac{1}{2} n + 200.$$

Subtract 200 from each side of the equation (2),

(3) $$225 = \tfrac{1}{2} n,$$

since, if 425 and $\frac{1}{2} n + 200$ are the same number, it follows that 425 less 200 is the same number as $\frac{1}{2} n + 200$ less 200. Now multiply each side of the equation (3) by 2,

$$450 = n,$$

since, if 225 and $\frac{1}{2} n$ are the same number, twice one of them is the same as twice the other. We now check this by putting $n = 450$ in the original equation (2); this gives

$$425 = \tfrac{1}{2}(450) + 200,$$

which is evidently correct.

The check used above is *complete* (p. 5); *i.e.* the check leaves no doubt whatever concerning the correctness of the answer. Whenever the answer can be tried directly in the given problem, as was done above, the check is complete. Hence, this should be done whenever possible.

35. Operations on Equations. The work just done is nothing more than we could have done in Chapter II, but it is *carefully* stated. In particular, we consciously *subtracted* 200 *from each side* of equation (2), and we *multiplied each side* of equation (3) by 2. The student will see that we may always do any one of the following things without disturbing the equality:

I. We may **add** the same number to each side of an equation.

II. We may **subtract** the same number from each side of an equation.

III. We may **multiply** each side of an equation by the same number.

IV. We may **divide** each side of an equation by the same number, when the division is possible (see p. 106).

For an equation indicates that *the two sides each represent the same number*, hence the result of *adding*, *subtracting*, *multiplying*, *or dividing* by the same number on both sides must be the same.* Always perform the operation upon each side **as a whole**, not upon a part of it.

36. Definition of Linear Equations. An equation with one unknown letter which contains the unknown number only in its first power, in the form: $Ax + B = 0$, where A and B are known numbers, or which can be reduced to that form by the operations of § 35, is called a **simple** or **linear** equation. Most of the equations we have studied up to this time are *linear*. (§ 20, p. 25.)

37. Examples. In this Chapter we have learned how to remove parentheses, how to add and subtract negative quantities as well as positive quantities. These operations assist in solving equations.

Ex. 1. Given the equation $2x - 4 = 5$, where x is an unknown number; to find x.

(1) $$2x - 4 = 5.$$

Add 4 to each side: $$2x = 5 + 4,$$

or, $$2x = 9.$$

---

* These rules really state the *axiom that there can be only one result* for any addition, or subtraction, or multiplication, or (possible) division. Notice that each of these operations is performed by means of a number; if any expression other than a simple number is used in any of them, care is necessary to insure that it does represent a number. See also p. 106.

Divide each side by 2: $x = \frac{9}{2} = 4\frac{1}{2}$.

Check: Put $4\frac{1}{2}$ for x in (1): $2(4\frac{1}{2}) - 4 = 5$, or $9 - 4 = 5$; since this is correct, we see that the value of x found is correct.

Ex. 2. Given the equation

(1) $$2(3x - 4) - 12 = 2[(x-1) - (2x-3)],$$

where x is unknown; to find x.

Remove the parentheses as in § 33, p. 53.

(2) $$6x - 8 - 12 = 2(x - 1 - 2x + 3),$$

or,

(3) $$6x - 20 = 2(-x + 2),$$

or,

(4) $$6x - 20 = -2x + 4.$$

Add $2x$ to each side:

(5) $$6x + 2x - 20 = 4.$$

Add 20 to each side: $6x + 2x = 4 + 20$,

or, $8x = 24$.

Divide each side by 8: $x = 3$.

Check: Put 3 in the place of x in (1),

$$2(3 \cdot 3 - 4) - 12 = 2[(3 - 1) - (2 \cdot 3 - 3)],$$

or, $$2(5) - 12 = 2(2 - 3),$$

or, $$10 - 12 = 2(-1),$$

or, $$-2 = -2,$$

which is seen to be correct: the answer $x = 3$ is therefore correct.

The general **plan of solution** is as follows:

(1) Perform all indicated operations. If necessary, remove fractional coefficients by III, § 35.

(2) Transpose all terms that contain the unknown letter whose value is required to one side; all other terms to the other side.

(3) Combine the terms on each side, and collect the coefficient of the unknown letter (§ 28).

(4) Divide both sides by the coefficient of the unknown letter.

(5) Check the answer thus found by trying it in the given equation.

EXERCISES IX: CHAPTER III

Solve the following equations to find the value of the unknown letter. If there are several, solve for the letter specified.

1. $x + 3 = 7$.

2. $x - 3 = -7$.

3. $2x + 5 = 17$.

4. $x - 3 = 7$.

5. $-x + 3 = 7$.

6. $9x - 7 = 6x + 8$.

7. $x + 3 = -7$.

8. $-x - 3 = -7$.

9. $2x + 5 = 4x - 1$.

10. $6x + 25 = x - 5$.

11. $6 - 5z = 3 - z$.

12. $7n - 16 = 3n - 2$.

13. $13p + 50 = 25 - (5 + 2p)$.

14. $12a + 1 - (3a - 4) = 2a + 8 + (4a + 4)$.

15. $6k + 9 = 2k - [4 - (10 + 2k)]$.

16. $2x + \frac{2}{3} = x + 2\frac{1}{3}$.

17. $\frac{1}{2}x + \frac{1}{3} = \frac{1}{3}x + \frac{1}{2}$.

18. $9r - 15 = 2r + 3$.

19. $\frac{7}{12}x - \frac{1}{4} = 2x - 1\frac{2}{3}$.

20. $3(x - 5) - 2(x - 4) = 0$.

21. $2(q - 2) + 4(1 - q) = 6$.

22. $76t - 139 = 196t + 57$.

23. $2y - 3 + 3(10 - 3y) = 3 - y$.

24. $\frac{3}{5}(n + 7) + \frac{1}{3}(n - 4) = -\frac{1}{4}n$.

25. $a^2 - ax = ab$. [Solve for x in terms of a and b.]

26. $\frac{7}{8}(z + 2) - \frac{2}{3}(12 - z) = \frac{1}{5}(z + 9)$.

27. $cz - c^2 = ac + bc$. [Solve for z; then solve for a; then for b.]

28. $vt = s$. [Solve for v; then solve for s; then for t.]

29. $ax + by + c = 0$. [Solve for x; also solve for y.]

30. $ax - by + c = a^2 - (by - c)$. [Solve for x.]

38. Transposition. In any equation any term upon one side may be removed from that side by subtracting it from both sides. This has been done before. The effect is the same as if we simply move the term to the other side of the equation *and change its sign.*

Thus, in Ex. 1, § 37, we moved 4 from the left side of equation (1) to the right side *with its sign changed.*

In Ex. 2, § 37, we moved $-2x$ from the right of equation (4) to the left, and changed the sign from $-$ to $+$.

This operation is often called **transposition**; *it consists in moving the term to the other side of the equation and changing its sign.* Hereafter we shall use it freely. The student must be careful to transpose only *terms, not factors.*

39. Problems stated in English. Examples stated in English may be put into algebraic language by choosing convenient letters to stand for quantities mentioned in the example. We have frequently done this (Chapter II).

Ex. 1. The sum of two numbers is 14, their difference is 2. Find the numbers.

Let n be the smaller number; then the other is $n+2$, since the difference is 2.

Hence, $n+(n+2)=14,$

or, $2n+2=14.$

Transpose 2: $2n=12.$

Divide by 2: $n=6$, the smaller number.

The larger number is $n+2=6+2=8$.

Check: the numbers being 6 and 8, their sum is $6+8=14$; their difference is $8-6=2$.

NOTE. This type of problem often arises in science and in business. For example, a boat may make 14 miles per hour going down a stream, but only 2 miles per hour going up the same stream. In that case the two numbers in our problem would be the *speed* of the boat in still water and the *speed* of the water in the stream; their sum is 14 (in miles per hour), their difference is 2 (in miles per hour). The work just done shows that the *speed* of the boat is 8 miles per hour, the *speed* of the stream 6 miles per hour.

Ex. 2. Two men enter a partnership, the first giving three times as much as the second. If they earn \$1000, what part should each receive?

Let $x=$ the share of the first partner, in dollars. Then $1000-x=$ the share of the second partner, in dollars. Since the first should receive three times what the second receives,

$$x=3(1000-x),\text{ or, } x=3000-3x.$$

Transposing $3x$, we get

$$4x = 3000.$$

Dividing by 4, we get

$x = 750$, the share of the first partner, in dollars.

Hence, $1000 - x = 250$, the share of the second partner, in dollars.

Check: $\$750 = 3 \times \$250$; $\$750 + \$250 = \$1000$.

Ex. 3. If two grades of coffee costing 25¢ and 35¢ are to be mixed so as to sell for 40¢ a pound at a profit of 25%, what parts should be taken to make 50 lb. of the mixture?

Let x = number of pounds of 25¢ coffee.
Then, $50 - x$ = number of pounds of 35¢ coffee.
Then the total cost of the 50 lb. is

$$25x + 35(50 - x), \text{ in cents.}$$

At a profit of 25% the total selling price of 50 lb. is

$$\tfrac{125}{100}[25x + 35(50 - x)].$$

If this is to sell at 40¢ a pound, the total received is $40 \times 50 = 2000$ (in ¢). Hence, $\tfrac{125}{100}[25x + 35(50 - x)] = 2000$,

or, $\tfrac{5}{4}[25x + 35(50 - x)] = 2000$.

Multiply both sides by 4:

$$5[25x + 35(50 - x)] = 8000.$$

Divide both sides by 5:

$$[25x + 35(50 - x)] = 1600,$$

or, $25x + 1750 - 35x = 1600$,

or, $1750 - 10x = 1600$.

Transpose 1600 and $10x$: $150 = 10x$.

Divide by 10: $15 = x$.

We have therefore 15 lb. of the 25¢ coffee, and $50 - 15 = 35$ lb. of the 35¢ coffee.

Check: The cost of the mixture is $15 \cdot 25 + 35 \cdot 35 = 1600$ (in cents). The selling price is therefore

$$1600 + 25\% \text{ of } 1600 = 1600 + 400 = 2000 \text{ (in cents).}$$

$2000 \div 50 = 40$ (in cents) is the selling price per pound. This is as desired; hence the answers found are correct.

EXERCISES X: CHAPTER III

The student will best proceed as follows:

(1) Read the problem carefully.

(2) Select one unknown quantity, and denote it by some letter.

(3) Express all unknown quantities in the problem in terms of the one just selected.

(4) State the fact given in the problem as an equation.

(5) Solve the equation by the method of § 37.

(6) Check the answer by trying it in the given problem.

1. Divide 126 into two parts, one of which is twice as great as the other.

2. Divide 32 into two parts, one of which is three times as great as the other.

3. Divide 75 into two parts which are to each other as 2 to 3.

4. One partner has three times as much invested in business as the other. How should a profit of $7000 be divided?

5. Two partners have invested in an enterprise $3000 and $2000. How should a profit of $360 be divided?

6. Three men engage in business; A contributes $1000, B $800, and C $750. A profit of $255 is realized. How should it be divided?

7. The sum of two numbers is 18, their difference is 6. What are the numbers?

8. The sum of two numbers is 9, their difference is 15. What are the numbers?

9. A ship sails against the current of a stream at 5 miles per hour, with the current at 20 miles per hour. What is the speed of the ship? of the current?

10. The sum of two numbers is s, their difference is d. What are the numbers?

From Ex. 10, find directly the two numbers whose

11. Sum is 16 and whose difference is 10.

12. Sum is 32 and whose difference is -4.

13. Sum is 17 and whose difference is 15.

14. Sum is -9 and whose difference is -5.

15. The difference between a number of two digits whose sum is 8, and the number formed by reversing the digits is 18. What is the number?

16. Find two consecutive integers whose sum is 31.

17. Show that the sum of two consecutive integers is an odd integer. If this sum is a, what are the integers?

Using Ex. 17, find two consecutive integers whose sum is:

18. 55. **19.** 13. **20.** 71. **21.** 45.

22. How can a merchant mix 10 pounds of tea worth 31 cents a pound out of tea worth 30 and 35 cents a pound?

23. A merchant mixes two grades of vinegar which cost him 55 cents a gallon and 65 cents a gallon, respectively. How much of each must he take to make a 100-gallon mixture which he can sell at 75 cents a gallon with a profit of 20 %?

24. Find three consecutive integers whose sum is 54.

25. Show that the sum of three consecutive integers is divisible by 3. If this sum is a, what are the integers?

26. What is the amount, if \$200 is lent at 5% simple interest for t years? In how many years will the amount be \$250?

27. What is the amount if \$$d$ is lent at 5 % interest for t years? When will the principal be doubled?

28. Solve example 27 for a rate of p %. (The answer will, of course, be in terms of the letters p, d, t.)

29. At what rate of simple interest will a sum of money be doubled in 25 years? in t years?

30. Find three consecutive even integers whose sum is 24.

31. Can you mention a number by which the sum of three consecutive even integers is divisible? *Ans.* 2, 3, or 6. (Student give the proof.)

32. What must be the sale price of a piece of land in order that, after deducting an agent's commission of 4 %, the seller may receive \$1200?

REVIEW EXERCISES XI: CHAPTER III

Perform the operations indicated in the following exercises, 1–6, regarding them first as additions, then as subtractions:

1.
$$\begin{array}{r} 6\,ax - 9\,by \\ -\,2\,ax - 3\,by \\ \hline \end{array}$$

2.
$$\begin{array}{r} -\,5\,x^3y^2 + 3\,x^2y^3 \\ -\,9\,x^3y^2 - x^2y^3 \\ \hline \end{array}$$

3.
$$\begin{array}{l} lx - my \\ mx + ly \\ \hline \end{array}$$

4.
$$\begin{array}{r} x^2 + 2\,xy + y^2 \\ 2\,x^2 + \;\; xy - y^2 \\ \hline \end{array}$$

5.
$$\begin{array}{r} n^4 + 2\,n^2 + 1 \\ n^4 - 2\,n^2 + 1 \\ \hline \end{array}$$

6.
$$\begin{array}{r} 2\,y + 7\sqrt{\dfrac{bz}{t}} \\ -\,2\,y + 7\sqrt{\dfrac{bz}{t}} \\ \hline \end{array}$$

7. How is the average of two numbers x and y found? If we know the average of x and y, how may we find the sum of x and y? If the average of x and y is 5, what is $x + y$? Plot the figure, showing all points, the average of whose coördinates is 5.

8. Plot the figure of all points such that the average of x and y is -3.

9. The average of $-x$ and y is -5. Plot the figure.

10. The average of x and $-3\,y$ is 2. Plot the figure.

11. The average of y and 2 is x. Plot the figure.

12. The average of $-5\,y$ and 10 is x. Plot the figure.

13. Denote by y the average of x and -10 for various values of x. What equation connects x and y? Make a table of values of x and y, and plot the figure.

14. Denote by y the average of $-3\,x$ and 6. Plot the figure.

15. Denote by y the average of $-2\,x$ and -3. Plot the figure.

Perform the operations indicated:

16. $(ax - 2by + 3cz) + (by - 2cz + 3ax) + (cz - 2ax + 3by)$.

17. $(6x^3y^2 + 3x^5 - 7xy^4) - (3xy^4 - 4y^5 - 7x^2y^3 + 2x^3y^2) + (x^5 - x^3y^2 + x^4y) - (4x^5 + 4y^5) + (10xy^4 - 7x^2y^3)$.

18. $(x^2 + y^2 + z^2 - 2yz - 2zx - 2xy) - [(y^2 + z^2 - 2yz) + (z^2 + x^2 - 2zx) + (x^2 + y^2 - 2xy) - (x^2 + y^2 + z^2)]$.

19. $(x^3 + y^3 + z^3 - 3xyz) - [(y^3 - 3y^2z + 3yz^2 - z^3) + (z^3 - 3z^2x + 3zx^2 - x^3) + (x^3 - 3x^2y + 3xy^2 - y^3)] + [(3y^2z - 3yz^2) + (3z^2x - 3zx^2) + (3x^2y - 3xy^2)]$.

Solve the following equations for the letters denoting unknown quantities:

20. $2(x-5) + 3(6-x) = 0$.

21. $6p + 4 - p = 19p + 25 - 7p$.

22. $\frac{1}{2}(z-5) + (10-z) = \frac{1}{3}z$.

23. $7 - [5 + (3 - 2n)] = 15$.

24. $(1 - w) - [w^2 - (3w - 5)] + [3 - (5 - w^2)] = 0$.

25. $(6k - 9) + [5 - (3k - 2)] = 0$.

26. $(2x + 7) - (5 + x) = -3$.

27. $2ax - x^2 = 2xy$. (Solve for a; then also solve for y.)

Simplify the expressions:

28. $[(6xy - 7ab) - 2(5xy - 3ab)] + [(5xy - 3ab) - 2(6xy - 7ab)]$.

29. $[2(10p^2 + 9pq - q^2) + 3(2p^2 - 3pq + 2q^2)] - [5(10p^2 + 9pq - q^2) - (2p^2 - 3pq + 2q^2)] + [2(10p^2 + 9pq - q^2) - 3(2p^2 - 3pq + 2q^2)]$.

30. The sum of two numbers is $3a + b$; their difference is $a - b$; what are the numbers?

31. The sum of two consecutive integers is $4k - 3$; what are the numbers?

32. How many pounds each of spice worth 20 and 50 cents a pound should be taken to form 12 pounds of a mixture worth 30 cents a pound?

33. What is the amount at compound interest at rate r on principal p for two years? What principal will yield \$605 at 10 per cent at compound interest for 2 years?

SUMMARY OF CHAPTER III: ADDITION AND SUBTRACTION; SIMPLE EQUATIONS, pp. 35–65

PART I. GENERAL RULES FOR OPERATION; PARENTHESES. pp. 34–54.

Extension of the Operations: Addition of a positive, forward motion; subtraction of a positive, backward motion. § 23, p. 34.

Fundamental properties of addition and multiplication: five rules:

I. $a+b=b+a$ (Commutative Law of Addition);

II. $a \times b = b \times a$ (Commutative Law of Multiplication);

III. $a+(b+c)=(a+b)+c$ (Associative Law of Addition);

IV. $a \times (b \times c) = (a \times b) \times c$ (Associative Law of Multiplication);

V. $a(b+c)=ab+ac$ (Distributive Law).

These five are *axioms.* Exercises I. § 24, pp. 34–36.

To add a negative number: backward motion.

To subtract a negative number: forward motion.

$\left\{\begin{matrix}\textit{Adding}\\ \textit{Subtracting}\end{matrix}\right\}$ *a negative number:* equivalence to $\left\{\begin{matrix}\textit{subtracting}\\ \textit{adding}\end{matrix}\right\}$ a *positive* number of the same amount.

To subtract any number: change its sign and add. § 25, pp. 36–37.

To add several numbers: the difference of negative total and positive total amounts with sign of greater total amounts. Exercises II. § 26, pp. 37–39.

Equivalent Expressions: $a+(-b)$ and $a-b$; $a-(-b)$ and $a+b$. Exercises III. § 27, pp. 39–44.

To add similar monomial terms: sum of coefficients × common factor; *to subtract:* change sign and add. Exercises IV. §§ 28, 29, pp. 44–45.

Use of parentheses: grouping of terms in addition. Exercises V. § 30, pp. 47–49.

To add longer expressions: add similar terms in columns. Exercises VI. § 31, pp. 49–51.

To subtract a longer expression: change the sign of each term and add. Exercises VII. § 32, pp. 51–53.

To remove or insert parentheses: change $\left\{\begin{matrix}\text{no}\\ \text{all}\end{matrix}\right\}$ signs if $\left\{\begin{matrix}+\\ -\end{matrix}\right\}$ precedes the parentheses. Exercises VIII. § 33, pp. 53–54.

CHAPTER IV. MULTIPLICATION AND DIVISION; FACTORING; APPLICATIONS

PART I. MULTIPLICATION AND DIVISION OF NUMBERS AND MONOMIALS

40. Multiplication. As in the case of addition, we wish to extend the idea of multiplication so that we can multiply any kinds of numbers together. We shall always keep the rules given on p. 35, and in every other way we shall try to preserve the spirit of what was known as multiplication in elementary arithmetic.

Multiplication was originally applied to integers only. In multiplying 5 by 4 we take 5 four times and add; thus, $5 \times 4 = 5 + 5 + 5 + 5 = 20$.

This notion very clearly does not hold for fractions. In multiplying 12 by $\frac{2}{3}$ we do *not* take 12 two thirds times, for that is absurd.

To multiply 12 by $\frac{2}{3}$ we may note that

$$12 \times \tfrac{2}{3} = \tfrac{2}{3} \times 12 = \tfrac{2}{3} + \tfrac{2}{3} + \cdots (12 \text{ times}) = 8,$$

by virtue of our Rule II, p. 35. In general, if a, b, c, are any positive integers,

$$a \times \frac{b}{c} = \frac{b}{c} \times a = \frac{b}{c} + \frac{b}{c} + \cdots (a \text{ times}) = \frac{ab}{c}.$$

If we now take, say $\frac{3}{5} \times \frac{4}{7}$, we note that

$$(\tfrac{3}{5} \times \tfrac{4}{7}) \times 35 = \tfrac{3}{5} \times (\tfrac{4}{7} \times 35) = \tfrac{3}{5} \times 20 = 12, \quad \text{by IV, p. 35.}$$

Hence, $\qquad (\frac{3}{5} \times \frac{4}{7}) \times 35 = 12,$

or, by IV, p. 56, $\qquad \frac{3}{5} \times \frac{4}{7} = \frac{12}{35}.$

Likewise, in general,

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}, \text{ for } \left(\frac{a}{b} \times \frac{c}{d}\right) \times bd = \frac{a}{b} \times \left(\frac{c}{d} \times bd\right)$$

$$= \frac{a}{b} \times cb.$$

$$= ac.$$

Thus, we arrive at the usual rule for multiplying fractions by means of the Rules II and IV, p. 35.

The elementary definition of multiplication (for integers) is: *The product of two integers is found by taking the multiplicand as many times as is indicated by the multiplier and adding.* We extend this definition for other numbers by saying: *The product of any two numbers shall be such that the rules of* p. 35 *shall remain true.*

Thus, the rule for fractions results in this way, as shown above. We shall use this principle to find the product in all new cases.

As another example, consider the product of any number and zero; say 4×0. We say

$$0 \times 4 = 0 + 0 + 0 + 0 = 0.$$

But, $4 \times 0 = 0 \times 4$ by Rule II; hence, $4 \times 0 = 0$.

The product of any number and zero is zero.

The proof shows this to be true for any integer only. The following argument shows that it is true also for any fraction:

To find $0 \times \frac{a}{b}$; consider $(c - c) \times \frac{a}{b} = 0 \times \frac{a}{b}$.

$$0 \times \frac{a}{b} = (c - c)\frac{a}{b} = \frac{c \cdot a}{b} - \frac{c \cdot a}{b} = 0.$$

This proof need not be learned at this time.

41. Products of Negatives and Positives. It frequently becomes necessary to multiply a negative number by a positive number.

Thus, if a man makes ten debts of \$50 each, his total debt is \$500, *i.e.* $(-50\,d)\times 10 = -500\,d$. This holds whether d means dollars or anything else.

In general, $(-a)\times b = -ab$.

To multiply a negative quantity by a positive quantity, multiply the two amounts as if both were positive and prefix the sign — to the answer.

This is convenient also in solving equations. Thus, the equation $3x-6=0$ gives $x=2$; check: $3(2)-6=0$ (correct). But if we first transpose $3x$, we get $-6=-3x$; an attempt to put 2 in place of x here gives $-6=-3(2)$, which must be correct in order to avoid contradicting the previous check.

Since the factors should give the same result when multiplied in any order, *the same rule holds for multiplying a positive number by a negative number.* In formula:

$$a\times(-b) = -ab.$$

We can easily *prove* that these results must be true if the rules on p. 35 hold true, but it is not necessary for the student to learn this proof at this time.

For $\quad a\times[b+(-b)] = a\times(b)+a\times(-b)$, by Rule V.

But, $\quad a\times[b+(-b)] = 0$ since $b-b=0$.

Hence, $\quad ab+a(-b)=0$,

or, $\quad a\times(-b) = -ab$, which is one of our rules.

Moreover, $(-b)\times a = a\times(-b)$ by Rule II, $=-ab$, which is the same as our rule above.

EXERCISES I: CHAPTER IV

1. A man owes twelve debts of \$30 each. What is his total indebtedness? State the problem in terms of negative numbers.

2. Each inhabitant of a city pays a certain tax of \$10. What is the increase in the wealth of the city due to the departure of fifteen residents?

3. How much is the downward pressure of a bar of iron lessened by the attachment of twelve springs, each pulling up

with a force of 75 pounds? Express in terms of negative numbers.

Perform the following multiplications:

4. 5×-2.	**9.** $3 \times (-1 \times 7)$.	**14.** $2 \times -(3x \times 2\frac{1}{2})$
5. 7×-3.	**10.** $7 \times -15d$.	**15.** 0×-3.*
6. 4×-1.	**11.** $6 \times -13k$.	**16.** -3×0.
7. -4×1.	**12.** $6 \times -5 \times 9kl$.	**17.** $-2\frac{1}{2} \times 0 \times 3x$.
8. -2×5.	**13.** $-9 \times 3x^2yz^3$.	**18.** $-7.2 \times 2.3x^2y$.

19. $-5 \times 6(bc+ca+ab)$. **20.** $-3(x^2+y^2-2xy) \times 10$.

21. $$-13\left(\frac{qr}{q-r}+\frac{rp}{r-p}+\frac{pq}{p-q}\right) \times 0.$$

22. In arithmetic we multiply by taking the multiplier as many times as the number of units in the multiplicand and adding. Supposing this to be a proper method whenever the multiplier is a positive integer, show that $-5 \times 12 = -60$; $0 \times 7 = 0$.

42. Negatives multiplied. It is frequently necessary in algebra *to multiply one negative number by another.* It is convenient, and, in fact, necessary if we are to avoid contradictions, to say that *the product is positive.*

Thus, given the equation $3x + 6 = 0$, we find $x = -2$; check: $3(-2)+6=0$ (correct). If we first transpose $3x$, we have $6 = -3x$; an attempt to put -2 in place of x in this form of the equation gives $6 = -3(-2)$, which must be correct if we are to avoid contradicting the previous check. Hence, we say $(-3)(-2) = 6$.

In general, $(-a) \times (-b) = +ab$, or,

To multiply one negative number by another, multiply their amounts as if both were positive and prefix the sign + to the answer.

---

* Observe that $+0 = -0$, since each means *no motion at all* on our scale. Hence, in the answers, write merely 0 whenever either $+0$ or -0 would result.

This is easily proved by the rules of p. 35. For

$$(-a)\times(b-b)=(-a)\times b+(-a)(-b),$$

or,
$$0=-ab+(-a)(-b),$$

or,
$$+ab=(-a)(-b).$$

[The student need not learn this proof at this time.]

The preceding rules may be stated as follows:

$$-a\times+b \text{ gives } -ab,$$
$$+a\times-b \text{ gives } -ab,$$
$$-a\times-b \text{ gives } +ab,$$

and, of course, $+a\times+b$ gives $+ab$, or,

RULE OF SIGNS: *In multiplying, like signs give + and unlike signs give −.*

EXERCISES II: CHAPTER IV

Perform the following multiplications:

1. $-2\times-3.$ **5.** $-7\times-5.$ **9.** $-13\times-5\,d.$

2. $-3\times7.3.$ **6.** $5\times-7.$ **10.** $-4\times-7\,xyz.$

3. $7\times-5.$ **7.** $-1\times-1.$ **11.** $7\times-5\times-.2.$

4. $-7\times5.$ **8.** $-13\times0.$ **12.** $-1\times-1\times-1.$

13. $2\times-3\,b^3yz\times-13.$ **14.** $-6\,kz\times-3\times-5.$

15. $-\frac{7}{8}\times\frac{1}{3}\frac{2}{5}(y^2+wx+xy).$ **16.** $-7\,x^2y\times0\times-3\frac{1}{2}.$

17. $-\dfrac{2}{9}\times-\dfrac{3}{8}\left(\dfrac{qr-st}{qs+rt}\right).$ **18.** $-9\times\dfrac{5}{3}\dfrac{kl}{m^3-n^2}.$

19. $-1\times-1\times-1\times-1.$ **20.** $-1\times-1\times-1\times-1\times-1.$

21. $(-3)^3.$ **22.** $(-3)^4.$ **23.** $(-3)^5.$

43. Multiplication of Monomials by Rearrangement. *Monomials may be multiplied by rearranging their factors.*

Thus,
$$\begin{aligned}4\,ab\times3\,ab &=(4\times a\times b)\times(3\times a\times b)\\ &=(4\times3)\times(a\times a)\times(b\times b)\\ &=12\times a^2\times b^2,\end{aligned}$$

since, by Rule II, p. 35, the order of multiplication is immaterial.

Check: If we put $a = 2$, $b = 3$, we find

$$4 \cdot 2 \cdot 3 \times 3 \cdot 2 \cdot 3 = 12 \cdot 2^2 \cdot 3^2 = 12 \cdot 4 \cdot 9 = 432 \text{ (correct)}.$$

Likewise, $(-4\,ab) \times (3\,ab) = -(4\,ab)(3\,ab) = -12a^2b^2$,

by the rule for signs given just above.

Check: If we put $a = 2$, $b = 3$, we find

$$(-4 \cdot 2 \cdot 3) \times (3 \cdot 2 \cdot 3) = -12 \cdot 2^2 \cdot 3^2 = -12 \times 4 \times 9 = -432 \text{ (correct)}.$$

EXERCISES III: CHAPTER IV

Perform the following multiplications by the method just given. Check each result by substituting numerical values for the letters.

1. $\begin{array}{r} -5\,a \\ 6\,b \\ \hline \end{array}$ **3.** $\begin{array}{r} -12\,x^2y \\ -3\,xy^2 \\ \hline \end{array}$ **5.** $\begin{array}{r} 7\,p \\ -8(k+m+n) \\ \hline \end{array}$

2. $\begin{array}{r} -2\,kr \\ -3\,x \\ \hline \end{array}$ **4.** $\begin{array}{r} 9\,uv^2w \\ -7\,xv^3u^2 \\ \hline \end{array}$ **6.** $\begin{array}{r} -2(x+y+z-1) \\ -3(x+y+z-1)^2 \\ \hline \end{array}$

7. $-2\,a \times -2\,a \times -2\,a$. **8.** $-7\,x^2y^3 \times 3\,xyz \times -kz^3$.

9. $(-3\,x)^3$. **10.** $(2\,m^2n)^2$. **11.** $(-3\,x^2yz)^4$.

12. $-(a+b)(x^2+y^2)^3 \times 3(x^2+y^2)(a+b)^2 \times -2(a+b)(x^2+y^2)$.

13. $l^2m^3n^4p^5q^6r \times -2\,l^3m^4n^5p^6qr^2 \times 3\,l^4m^5n^6pq^2r^3 \times -4\,l^5m^6np^2q^3r^4$.

44. Multiplication of Simple Powers. By the same rule we may multiply any simple powers (§ 9, p. 8) of the same quantity.

Thus, $2^2 \times 2^3 = (2 \times 2) \times (2 \times 2 \times 2) = 2 \times 2 \times 2 \times 2 \times 2 = 2^5$,

and, $a^2 \times a^3 = (a \times a) \times (a \times a \times a) = a \times a \times a \times a \times a = a^5$,

no matter what a may mean. Likewise,

$$a^4 \times a^3 = a \cdot a \cdot a \cdot a \times a \cdot a \cdot a = a^7.$$

In general, for simple powers,

$$a^m \times a^n = a \cdot a \cdots (m \text{ times}) \times a \cdot a \cdots (n \text{ times})$$
$$= a \cdot a \cdot a \cdots (m+n \text{ times}) = a^{m+n}.$$

This rule is proved now only for simple powers, *i.e.* when m and n are positive integers. Later we shall prove that it holds for other values of m and n. See Chapters VII and XI.

Note that the exponent in the product is the **sum** *of the given exponents.*

If this rule is forgotten, it will speedily be found again by actually writing out the meaning of the separate factors, as above; but this rule is simpler. It is well to *remember* the example given above,

$$a^2 \times a^3 = a^5,$$

in order to remember the general rule.

Do not forget that a quantity with no exponent expressed really has an exponent *unity*,

$$a = a^1.$$

45. Final Rule; Monomials. By using the above rule we may multiply more simply.

Thus, $(4\,a^2b^2) \times (-2\,ab^3) = (4 \cdot -2)\,(a^2 \cdot a)\,(b^2 \cdot b^3) = -8\,a^3b^5.$

Do not fail to choose the proper sign for the product according to the rule of signs given above.

Several factors may be multiplied together at once, if care is used.

Thus, $(4\,a^2b^2) \times (-2\,ab^3) \times (3\,ab) = -24\,a^4b^6.$

The coefficient in the product is the product of the coefficients in the given factors; the exponent of any letter in the product is the sum of the exponents of that letter in the given factors.

If this rule is forgotten, the same problems can all be done by § 43.

The product of any number of positive factors is positive. The product of an even number of negative factors is positive; of an odd number of negative factors, negative. The product of any number of positive and negative factors is positive when the number of negative factors is even, and negative when the number of negative factors is odd.

EXERCISES IV: CHAPTER IV

Perform the following multiplications by the method just explained; do the first five also by the method of § 43. Check each result by substituting numerical values.

1. $-3\,a^2x \times -2\,ax^2y.$

2. $-3\,zc^2 \times 2\,z^2c^3.$

3. $7\,gt \times -\frac{1}{2}\,t.$

4. $9(2^2 \times 3^3) \times 3(2 \times 3^2).$

5. $-3\,bc \times -ca \times -ab.$

6. $6\,p^2q^3r^{10} \times -3\,pq^2 \times -2\,rp_3.$

7. $-4\,ab^2c \times -3\,a^2bc \times -2\,abc^2.$

8. $2.5\,m^2n^3 \times -8\,m^3p^5 \times +\frac{1}{2}\,n^4p^2.$

9. $5\,f^2g^2h^3 \times -3\,fg^3h \times -2\,f^2h^3.$

10. $7.2\,a^3bc^2 \times -4.1\,ab^3c \times 1.3\,a^2bc^3.$

11. $-3\,(y-z)(z-x) \times 2\,(z-x)(x-y)^2 \times -(y-z).$

12. $-2(-3\,bc)(-2\,ca) \times -3(-2\,ca)(-ab) \times (-ab)(-3\,bc).$

13. $5\sqrt{hg} \times -3(\sqrt{hg})^3 \times -2\,(\sqrt{hg})^5.$

14. $-18\left(\dfrac{mn}{p}\right)\left(\dfrac{xy}{a-b}\right)^5 \times \dfrac{1}{6}\left(\dfrac{mn}{p}\right)^2\left(\dfrac{xy}{a-b}\right) \times -\dfrac{1}{2}\left(\dfrac{xy}{a-b}\right)^2.$

Simplify the result as much as possible.

15. $a^m \times a^n.$ **16.** $-a^m \times -a^n.$ **17.** $(-a)^m \times (-a)^n.$

18. $(-a)^2;\ (-a)^3;\ (-a)^4;\ (-a)^5;\ (-a)^6;\ (-a)^7.$

19. $(-a)^m$, if m is even. **20.** $(-a)^m$, if m is odd.

21. $(-2\,x)^3.$ **22.** $(3\,rs^2t^3)^2.$ **23.** $(-4\,x^2y)^3(2\,xy^2)^4.$

46. Division. Division is the reverse of multiplication; the result of division is called the quotient; that is, *dividend* $\div$ *divisor* $=$ *quotient*, if *quotient* $\times$ *divisor* $=$ *dividend*.

For example:

$$12 \div 3 = 4, \text{ for } 4 \times 3 = 12.$$

Again, $\frac{3}{4} \div \frac{2}{5} = \frac{15}{8}$, for $\frac{15}{8} \times \frac{2}{5} = \frac{3}{4}$.

In general, $\dfrac{a}{b} \div \dfrac{c}{d} = \dfrac{ad}{bc}$, for $\dfrac{ad}{bc} \times \dfrac{c}{d} = \dfrac{a}{b}$.

We shall consider this again later (Chap. V, p. 134).

A negative number divided by a positive number gives a negative result; thus,

$$-12 \div 3 = -4, \text{ for } -4 \times 3 = -12.$$

A positive number divided by a negative number gives a negative result; thus,

$$(+12) \div (-3) = -4, \text{ for } -4 \times -3 = 12.$$

A negative number divided by a negative number gives a positive result; thus,

$$(-12) \div (-3) = +4, \text{ for } 4 \times -3 = -12.$$

The rule may be stated in short as follows:

RULE OF SIGNS. *In division, like signs give plus; unlike signs give minus.*

Notice that this rule *reads precisely like* the rule for signs in multiplication, p. 71. Let the student show *why* this *should* be the case.

Zero divided by any number, not zero, gives zero; thus, $0 \div 3 = 0$, for $0 \times 3 = 0$ (p. 68).

The quotient of any number divided by zero does not exist; thus, $3 \div 0 = ?$ (does not exist), for the question, $? \times 0 = 3$, has no answer.

EXERCISES V: CHAPTER IV

Perform the following divisions:

1. $-18 \div -6$. **3.** $18 \div -6$. **5.** $-8 \div -2$. **7.** $-5 \div 2$.

2. $-18 \div 6$. **4.** $15 \div -3$. **6.** $7 \div -3$. **8.** $-13 \div -5$.

9. $6\,a \div -2\,a$. **10.** $15\,bcp \div -3$. **11.** $-17\,xyz \div -2\,xyz$

12. $-3(bc+ca+ab) \div -2$. **13.** $-\dfrac{5\,gt^2}{2} \div -\dfrac{gt^2}{2}$.

14. $16\left(\dfrac{m^2-M^2}{Rw}\right) \div -7\left(\dfrac{m^2-M^2}{Rw}\right)$.

15. $-abcxyz \div -abc$. **16.** $0 \div -6\,x^2yz^3$. **17.** $-51\,m \div 0$.

47. Division of Simple Powers. We may find out how to divide many other expressions by the same reasoning, when the exponents are positive integers.

Thus, $a^5 \div a^3 = a^2$, because $a^2 \times a^3 = a^5$,

and $a^7 \div a^4 = a^3$, because $a^3 \times a^4 = a^7$,

and $a^m \div a^n = a^{m-n}$ because $a^{m-n} \times a^n = a^m$.

This rule is proved now only when m and n are positive integers, and m is greater than n. Later we shall prove that it holds for other values of m and n. See Chapters VII and XI.

Hence, in dividing powers of the same quantity, the exponent in the quotient is the *difference* of the exponents, the exponent in the divisor being subtracted from that in the dividend.

The rule is seen also by writing out the factors in full:

$$a^7 \div a^4 = \frac{a \cdot a \cdot a \cdot \cancel{a} \cdot \cancel{a} \cdot \cancel{a} \cdot \cancel{a}}{\cancel{a} \cdot \cancel{a} \cdot \cancel{a} \cdot \cancel{a}} = a \cdot a \cdot a = a^3,$$

as above. If the exponent of the divisor is the greater, the quotient is naturally a fraction. Thus,

$$a^4 \div a^7 = \frac{\cancel{a} \cdot \cancel{a} \cdot \cancel{a} \cdot \cancel{a}}{\cancel{a} \cdot \cancel{a} \cdot \cancel{a} \cdot \cancel{a} \cdot a \cdot a \cdot a} = \frac{1}{a \cdot a \cdot a} = \frac{1}{a^3}.$$

These divisions can be carried out always by carefully writing down the *meaning* of each factor, as in this example.

It should be noticed that

$$\frac{a^3}{a^3} = 1; \text{ for } \frac{a^3}{a^3} = \frac{\cancel{a} \cdot \cancel{a} \cdot \cancel{a}}{\cancel{a} \cdot \cancel{a} \cdot \cancel{a}} = 1.$$

In general, the quotient of two identical powers of the same quantity is unity.

48. Division of Monomials. To divide one monomial by another we merely apply the rules above. For example:

$$\frac{12\, a^5b^4c^3}{3\, a^2b^2c^3} = 4\, a^3b.$$

This may also be done as follows:

$$\frac{12\,a^5b^4c^3}{3\,a^2b^3c^3}=\frac{4\times 3\times \cancel{a}\cdot\cancel{a}\cdot a\cdot a\cdot a\times b\cdot\cancel{b}\cdot\cancel{b}\cdot\cancel{b}\times\cancel{c}\cdot\cancel{c}\cdot\cancel{c}}{\cancel{3}\times\cancel{a}\cdot\cancel{a}\times\cancel{b}\cdot\cancel{b}\cdot\cancel{b}\times\cancel{c}\cdot\cancel{c}\cdot\cancel{c}}=4\,a^3b\cdot$$

The cancellation is done upon the same principle as in arithmetic: the numerator (dividend) and the denominator (divisor) may each be divided by the same quantity without altering the value of the fraction (quotient).

As a convenient rule we notice that *the* **coefficient** *in the quotient is the* **quotient** *of the given coefficients; the* **exponent** *of any letter in the quotient is the* **difference** *of the given exponents of that letter in the order mentioned above.*

Remember the *rule of signs:* the quotient is positive or negative according as dividend and divisor have like or unlike signs.

EXERCISES VI: CHAPTER IV

Perform the following divisions; check each result by substitution of numerical values:

1. $-15\,ab^2c \div -5\,bc$.

2. $-5\,x^3y^5 \div 3\,xy^2$.

3. $12\,x^3y^{10} \div -4\,xy^2$.

4. $-5\,ab^2x \div -ab^2x$.

5. $3\,a^2b^3c^3 \div -2\,abc$.

6. $-15\,lm^3n^2 \div -5\,l^2mn^3$.

7. $-6\,kz^3 \div 3\,kr^2z$.

8. $-17\,klv \div -5\,k^2l^3v^3$.

9. $a^mb^nc^p \div -a^rb^sc^t$.

10. $(-1)^6 \div -(-1)^3$.

11. $(-a)^5 \div -a^3$.

12. $(-a)^4 \div a^3$.

13. $-a^4 \div -a^3$.

14. $0 \div -5\,p^2q^3r^{12}$.

15. $-5\,p^2q^3r^{12} \div 0$.

16. $(x+y)^3(a+b)^2 \div (x+y)^2(a+b)$.

Perform the indicated operations, expressing the results in simplest form:

17. $\dfrac{12\,a^2b \times -3\,ab^2}{-6\,ab}$.

18. $\dfrac{-13\,mn^2p^3 \times -5\,mp}{10\,m^3n^3}$.

19. $\dfrac{A^3a^lb^n \times -B^ra^lb^n}{-ABab}$.

20. $\dfrac{-14\,xyz \times 3\,x^2y^3 \times -10\,yz^5}{-21\,x^3y^3z^3}$.

21. $\dfrac{ax^ly^mz^n \times bx^{n-l}y^{l-m}z^{m-n}}{abx^{n-m}y^{l-m}}$.

22. $\dfrac{(x+y)^3(a+b)^2 \times (x+y)^2(a+b)^3}{2\,(x+y)^2\,(a+b)^2}$.

PART II. MULTIPLICATION AND DIVISION OF LONGER EXPRESSIONS

49. Monomial × Binomial. In § 28, p. 44, we saw that

$$9d + 3d = (9 + 3)d,$$

and so on. We may now multiply any binomial by a monomial by the same prineiple.

Thus,
$$\begin{aligned}(3a^2 + 2b^2)4ab &= (3a^2)(4ab) + (2b^2)(4ab)\\ &= 12a^3b + 8ab^3.\end{aligned}$$

The product of a monomial and a binomial is the sum of the products of the monomial times each of the terms of the binomial.

This is really merely a restatement of Rule V, p. 35:

$$a(b + c) = ab + ac.$$

Care should be taken not to overlook negative signs.

In checking such problems, it is often sufficient to *set each letter equal to unity.* In the preceding example, if $a = 1$ and $b = 1$, $(3a^2 + 2b^2)4ab = (3 + 2)4 = 12 + 8 = 20$, and $12a^2b + 8ab^3 = 12 + 8 = 20$ (correct). This check is not complete, but it serves to convince us that the work is correct. Try other numbers.

50. Monomial × Longer Expressions. The product of a monomial and any longer expression is found by the same process; for example:

$$\begin{aligned}4ab[2a^2 - 3ab + 5b^2] &= (4ab)(2a^2) + (4ab)(-3ab)\\ &\qquad + (4ab)(5b^2)\\ &= 8a^3b - 12a^2b^2 + 20ab^3.\end{aligned}$$

The work may be arranged as follows:

Multiplicand:	$2a^2 - 3ab + 5b^2$
Multiplier:	$4ab$
Product:	$8a^3b - 12a^2b^2 + 20ab^3$

Check: If we set $a = 1$ and $b = 1$ as in § 49, we get

$$4ab[2a^2 - 3ab + 5b^2] = 4[2 - 3 + 5] = 16;$$

and $\quad 8a^3b - 12a^2b^2 + 20ab^3 = 8 - 12 + 20 = 16$ (correct).

Care should be taken not to overlook negative signs.

EXERCISES VII: CHAPTER IV

Perform the multiplications indicated; check each result by substitution of numerical values:

1. $\begin{array}{l} x+y \\ \underline{a} \end{array}$

2. $\begin{array}{l} x-y \\ \underline{a} \end{array}$

3. $\begin{array}{l} x-y \\ \underline{-a} \end{array}$

4. $\begin{array}{l} -x-y \\ \underline{-a} \end{array}$

5. $\begin{array}{l} x+y \\ \underline{-2} \end{array}$

6. $\begin{array}{l} 2x-3 \\ \underline{5} \end{array}$

7. $\begin{array}{l} -3x+7 \\ \underline{-2} \end{array}$

8. $\begin{array}{l} y-x-9 \\ \underline{-3} \end{array}$

9. $\begin{array}{l} ax-by \\ \underline{2ax} \end{array}$

10. $\begin{array}{l} mn^2-m^2n \\ \underline{3mn} \end{array}$

11. $\begin{array}{l} x^2-2xy+y^2 \\ \underline{xy} \end{array}$

12. $\begin{array}{l} 6x^3y-3xyz^2 \\ \underline{2yz^3} \end{array}$

13. $\begin{array}{l} a^2b^3c-3abc^3 \\ \underline{-ab^2} \end{array}$

14. $\begin{array}{l} 7x^2-2x+4 \\ \underline{-3x^3} \end{array}$

15. $\begin{array}{l} ax-by-cz \\ \underline{-abxy} \end{array}$

16. $\begin{array}{l} -3x^4+6x^3-4x^2+5x-1 \\ \underline{-2x^2} \end{array}$

17. $\begin{array}{l} (x-y)^2+3(y-z)^2+(z-x)^2 \\ \underline{(x-y)^3} \end{array}$

18. If one fifth of y is known, how may y be found? If $\frac{y}{5}=-2$, $y=$? If $\frac{y}{5}=x$, $y=$? If $\frac{y}{5}=x-w$, $y=$?

19. Plot the figure for the equation $\frac{y}{3}=x-1$.

51. Division by Monomials. Monomial Factors. *To divide an expression by a monomial divide each term by the monomial and connect these partial results by the proper signs.*

Thus, $(8a^3b-12a^2b^2+20ab^3)\div(4ab)=$

$$\frac{8a^3b}{4ab}+\frac{-12a^2b^2}{4ab}+\frac{20ab^3}{4ab}=2a^2-3ab+5b^2.$$

The truth of this rule is evident because *quotient* × *divisor* = *dividend*: thus, the example just solved is the example of § 50 reversed.

Care should be taken not to overlook negative signs. In checking divisions by substituting numbers for letters, be careful to avoid division by zero (p. 75). If zero occurs, try other numbers.

This rule is chiefly useful in finding *monomial factors* of expressions. Thus, given the expression $8a^3b - 12a^2b^2 + 20ab^3$, we can see by inspection that $4ab$ is a factor, and we write

$$8a^3b - 12a^2b^2 + 20ab^3 = 4ab(2a^2 - 3ab + 5b^2).$$

[Let the student read again and state the definition of factor, § 8, p. 8.]

EXERCISES VIII: CHAPTER IV

Perform the following divisions; check each result by substitution of numerical values:

1. $\dfrac{kx - 3k}{k}$. **2.** $\dfrac{6a^2 - 9ab}{-3a}$. **3.** $\dfrac{-12xy - 16ax + 3x}{-4x}$.

4. $\dfrac{a^2b^3 - 5a^3b^2 - a^2b^2}{-a^2b^2}$. **7.** $\dfrac{s^2 - as - bs - cs}{3s}$.

5. $\dfrac{2fx^2 + 3gxy - 7hxz}{2x}$. **8.** $\dfrac{9ax^3 - 12a^2x^5 - 27a^3x^2}{-3ax^2}$.

6. $\dfrac{\frac{1}{2}gt^2 - \frac{7}{4}g}{\frac{1}{4}g}$. **9.** $\dfrac{12x^2y^3z^5 - 9x^5y^2z^3 + 15x^3y^5z^2}{3x^2y^2z^2}$.

Factor the following expressions as the product of a monomial factor and another expression:

10. $ab + ac$. **16.** $2ax^3 - 6a^2x^2 + 8a^3x$.

11. $4xy + 6xz$. **17.** $15m^3n^2 - 20m^2n^3 + 25m^3n^3$.

12. $5a^2b^3 - 10a^3b^2$. **18.** $8ab^2c^3 - 12a^2b^3c + 10a^3b^2c$.

13. $6m^3n^5 - 8m^5n^3$. **19.** $6r^3s^4t^2 + 12r^4s^2t^3 - 18r^2s^3t^4$.

14. $-9ar^6 - 12a^2r^2$. **20.** $8ap^3q^5 + 24bp^2q^3 - 32p^3q^2$.

15. $-11apq^2 + 13ap^2q$. **21.** $-10x^2yz - 5xy^2z - 15xyz^2$.

22. Factor the numerator in each of the exercises 1–9.

23. Express the *answer* in each of the exercises in Ex. VII as the product of two factors.

24. If twice a number is known, how is the number found? If $2x = 12$, $x = ?$ If $2x = -8$, $x = ?$ If $ax = a^2$, $x = ?$

25. If $ax = a^2 - 2ay$, $x = ?$ If $ax = a^2 + ab + ay$, $x = ?$

26. If $2x-3=5$, $x=$? If $ax-ab=ay$, $x=$?

27. $12x+16y=48$. Reduce to simpler form and plot the figure both before and after this reduction.

Simplify:

28. $(ax^m+bx^n)\div x^r$.

29. $(ax^my^n+bx^ky^l)\div x^ry^s$.

30. $(x^ny^n-2xy)\div xy$.

31. $(5x^{l-n}-15x^{m-n})\div -5x^{a-n}$.

52. Product: Two Binomials. In the product of two binomials, for example, $(a+b)(c+d)$, we may write first

$$(a+b)(c+d)=a(c+d)+b(c+d)$$

by what precedes, if we regard $(c+d)$ as a simple quantity.

Now, $$a(c+d)=ac+ad,$$

and $$b(c+d)=bc+bd.$$

Hence, $$(a+b)(c+d)=ac+ad+bc+bd.$$

Notice that this is true for numbers of various kinds:

$$(4+2)(3+6)=(4+2)\cdot 9=4\cdot 9+2\cdot 9.$$

But, $$4\cdot 9=4(3+6)=4\cdot 3+4\cdot 6,$$

$$2\cdot 9=2(3+6)=2\cdot 3+2\cdot 6.$$

Hence, $$(4+2)(3+6)=4\cdot 3+4\cdot 6+2\cdot 3+2\cdot 6 \text{ (correct).}$$

As an example, consider $(2ab+3b^2)(5a-4b)$.

We write this in the form:

Multiplicand:	$2ab+3b^2$
Multiplier:	$5a-4b$
First partial product:	$10a^2b+15ab^2$
Second partial product:	$-\ 8ab^2-12b^3$
Total product:	$10a^2b+\ 7ab^2-12b^3$

Care must be taken not to overlook negative signs.

In writing down the partial products we write in one line the product $(2ab+3b^2)(5a)$, and then the product $(2ab+3b^2)(-4b)$. The student should be careful to write *similar terms* in these products underneath one another, so that they shall be conveniently placed for addition.

53. Product of any Expressions. The product of longer expressions may be found by a similar rule. In any case we multiply each term of one of the expressions by each term of the other in some convenient order and add the resulting partial products.

The work will be easier if the terms in each of the given expressions are arranged first in a definite order; thus, if some letter is selected that occurs in most of the terms, *the terms may be arranged so that the exponents of that letter increase as we go toward the right or left.*

Thus, to multiply, $+4x - 3x^2 + 2$ by $2 + 5x^2 - 3x$,

we rearrange with respect to the letter x and write:

Multiplicand:	$2 + 4x - 3x^2$
Multiplier:	$2 - 3x + 5x^2$
First partial product:	$4 + 8x - 6x^2$
Second partial product:	$\quad - 6x - 12x^2 + 9x^3$
Third partial product:	$\quad\quad + 10x^2 + 20x^3 - 15x^4$
Product:	$4 + 2x - 8x^2 + 29x^3 - 15x^4$

Check: If $x = 1$, multiplicand $= 3$; multiplier $= 4$; product $= 12$ (correct).

An arrangement in which the exponents of the chosen letter increase as we go toward the left is equally good. In either arrangement the similar terms fall under one another if we shift the partial products to the right by one term each, provided all positive integral powers up to the highest are present in *both* factors. If any terms we should naturally expect are absent, care should be taken to put similar terms in the partial products under one another.

EXERCISES IX: CHAPTER IV

Perform the following multiplications, and check as usual:

1. $\begin{array}{c} x+y \\ \underline{a+b} \end{array}$	**4.** $\begin{array}{c} x-y \\ \underline{-a-b} \end{array}$	**7.** $\begin{array}{c} a+b \\ \underline{a-b} \end{array}$	**10.** $\begin{array}{c} a^2-ab+b^2 \\ \underline{a+b} \end{array}$
2. $\begin{array}{c} x-y \\ \underline{a+b} \end{array}$	**5.** $\begin{array}{c} a+b \\ \underline{a+b} \end{array}$	**8.** $\begin{array}{c} 2x-3y \\ \underline{-x+5y} \end{array}$	**11.** $\begin{array}{c} a^2-ab+b^2 \\ \underline{a-b} \end{array}$
3. $\begin{array}{c} x-y \\ \underline{a-b} \end{array}$	**6.** $\begin{array}{c} a-b \\ \underline{a-b} \end{array}$	**9.** $\begin{array}{c} x+2y \\ \underline{2x-y} \end{array}$	**12.** $\begin{array}{c} a^2+ab+b^2 \\ \underline{a+b} \end{array}$

13. $\begin{array}{l} a^2+ab+b^2 \\ \underline{a-b} \end{array}$

14. $\begin{array}{l} x^2+3x-4 \\ \underline{2x+3} \end{array}$

15. $\begin{array}{l} -m^4+2m^2-7 \\ \underline{2m^2-3} \end{array}$

16. $\begin{array}{l} r^3+5r-1 \\ \underline{r^2+3r+4} \end{array}$

17. $\begin{array}{l} x^2y^2-3xy+4 \\ \underline{-3xy+5} \end{array}$

18. $\begin{array}{l} 1-4x+x^2 \\ \underline{3+3x-2x^2} \end{array}$

19. $\begin{array}{l} x^2-3xy+2y^2 \\ \underline{3x-2y} \end{array}$

20. $\begin{array}{l} x^2+5-3x \\ \underline{2x-6+x^2} \end{array}$

21. $\begin{array}{l} p^3-2pq^2+7q^3 \\ \underline{p^2+pq+q^2} \end{array}$

22. $\begin{array}{l} 2x^5-3x^3-x^2+7x-3 \\ \underline{2x-1} \end{array}$

23. $\begin{array}{l} a^3x^2-2a^2x^3-3ax^4+x^5 \\ \underline{-a^2+ax-2x^2} \end{array}$

24. $\begin{array}{l} 2x^3-7x^2-4x+2 \\ \underline{x^2-5x+1} \end{array}$

25. $\begin{array}{l} 4r^3-6r^2+r+1 \\ \underline{-r^2+3r+2} \end{array}$

26. $\begin{array}{l} 3m^4-2m^3+6m^2+4m-3 \\ \underline{2m^3-2m+3} \end{array}$

27. $\begin{array}{l} -2x^4+3x^2-5 \\ \underline{x^3+x^2+x+1} \end{array}$

28. $\begin{array}{l} 1-2x^2+3x^3-4x^4 \\ \underline{1-x^2+2x^3} \end{array}$

29. $\begin{array}{l} x^3+2-3x^2+x-4x^4 \\ \underline{x^2+1-3x} \end{array}$

30. $\begin{array}{l} r^4-3r^3t+2r^2t^2-rt^3+t^4 \\ \underline{4r^2-t^2+rt} \end{array}$

Find the following products and powers:

31. $(x-1)(x-2)(x-3)$.

32. $(x-a)(x+a)(x^2+a^2)$.

33. $(a-5b)(a+3b)(a-b)$.

34. $(a+b)^2$.

35. $(a+b)^3$.

36. $(a+b)^4$.

37. $(a-b)^2$.

38. $(a-b)^3$.

39. $(a-b)^4$.

40. $(a+b+c)(a^2+b^2+c^2-bc-ca-ab)$.

41. $(x^2-4)(x^2+4)$.

42. $(x^l+y^m)(x^l-y^m)$.

Show that:

43. $(x^2+xy+y^2)(x^2-xy+y^2)=x^4+x^2y^2+y^4$.

44. $(a^2+b^2)(x^2+y^2)-(ax+by)^2=(ay-bx)^2$.

54. Division of Longer Expressions. Just now we shall not attempt to divide one expression by another except in a few easy cases in which the quotient is known to be simple. This is the case if the dividend is really known to be the product of the divisor times some simple expression.

Thus, taking the example worked out in § 53, we know that we can divide $4 + 2x - 8x^2 + 29x^3 - 15x^4$ by $2 + 4x - 3x^2$, and we know that the quotient is $2 - 3x + 5x^2$ because we just multiplied the latter two expressions and found their product to be the first expression here mentioned. If the quotient were unknown, we should write down the following scheme (see explanation below):

		Divisor
Dividend:	$4 + 2x - 8x^2 + 29x^3 - 15x^4$	$2 + 4x - 3x^2$
		$2 - 3x + 5x^2$
1st Partial Product:	$\underline{4 + 8x - 6x^2}$ (subtract)	*Quotient*
1st Remainder:	$-6x - 2x^2 + 29x^3 - 15x^4$	
2d Partial Product:	$\underline{-6x - 12x^2 + 9x^3}$ (subtract)	
2d Remainder:	$10x^2 + 20x^3 - 15x^4$	
3d Partial Product:	$\underline{10x^2 + 20x^3 - 15x^4}$ (subtract)	
Final Remainder:	0	

Check: If $x=1$, dividend $= 12$; divisor $= 3$; quotient $= 4$ (correct).

The final remainder being zero, the division is said to be **exact**.

This scheme is useful in rediscovering the partial products in the work in the example of § 53.

The **explanation** is as follows:

(1) *The first term of the dividend divided by the first term of the divisor gives the first term of the quotient.* Otherwise the work of § 53 would not give the first term of the product as shown.

(2) *This first term of the quotient × the divisor is the first partial product in § 53: we place it underneath the dividend.*

(3) *The difference between the dividend and the first partial product must be all the rest of the whole product in § 53; we therefore subtract them* (result called "1st remainder").

The next steps are all taken for similar reasons and are really repetitions of the above steps; they are:

(4) (*First term* of *first remainder*) ÷ (*first term* of *divisor*) = (*second term* of *quotient*)

(5) (Second term of quotient) × (divisor) = (second partial product).

(6) (First remainder) − (second partial product) = (second remainder).

(7) (First term of second remainder) ÷ (first term of divisor) = (third term of quotient, *last in this example*).

(8) (Third term of quotient) × (divisor) = (third partial product).

(9) (Third remainder) − (third partial product) = *final remainder* (= 0 *in this example*).

Steps (1), (2), (3) are simply *repeated* as many times as necessary. Thus, the steps (4), (5), (6) and the steps (7), (8), (9) are the same kind of steps.

If the **final remainder is zero**, the division is said to be **exact**; this always happens if the dividend is really the quotient multiplied by another simple expression.

If the remainder is not zero, the division cannot be entirely carried out; in that case what is left over is called the **final remainder**, or simply the **remainder**. This will usually happen if the example has not been carefully selected to avoid it.

Thus, if we try to divide $6 - 10x + 3x^2 - 7x^3 - 24x^4$ by $2 + 4x - 3x^2$, the work is as follows:

$$\begin{array}{llll}
\textit{Dividend:} & 6 - 10x + 3x^2 - 7x^3 - 24x^4 & \left| \; 2 + 4x - 3x^2 \right. & \textit{Divisor} \\
\textit{1st p. p.} & \underline{6 + 12x - 9x^2} \quad \text{(subtract)} & \left| \; 3 - 11x + 28x^2 \right. & \textit{Quotient} \\
\textit{1st rem.} & \quad -22x + 12x^2 - 7x^3 - 24x^4 & & \\
\textit{2d p. p.} & \quad \underline{-22x - 44x^2 + 33x^3} \quad \text{(subtract)} & & \\
\textit{2d rem.} & \qquad\qquad 56x^2 - 40x^3 - 24x^4 & & \\
\textit{3d p. p.} & \qquad\qquad \underline{56x^2 + 112x^3 - 84x^4} \quad \text{(subtract)} & & \\
 & \qquad\qquad\qquad -152x^3 + 60x^4 \quad \textit{Final Remainder} & &
\end{array}$$

In this case the division is "not exact" since the final remainder is not zero. Just as in arithmetic, we may still express the result; thus $31 \div 7$ gives quotient 4 and remainder 3; we say

$$\textit{dividend} \div \textit{divisor} = \textit{quotient} + \frac{\textit{remainder}}{\textit{divisor}},$$

i.e. $31 \div 7 = 4 + \frac{3}{7}$. Similarly, here we say

$$\underset{\text{(dividend)}}{(6 - 10x + 3x^2 - 7x^3 - 24x^4)} \div \underset{\text{(divisor)}}{(2 + 4x - 3x^2)}$$

$$= \underset{\text{(quotient)}}{(3 - 11x + 28x^2)} + \frac{\overset{\text{(remainder)}}{-152x^3 + 60x^4}}{\underset{\text{(divisor)}}{2 + 4x - 3x^2}}.$$

It is absolutely necessary for all this work that both dividend and divisor be carefully arranged in the same order (either exponents increasing to the left or to the right) *with regard to the same letter.*

In the examples just worked *the exponents increase to the right;* we may work the same examples equally well with *exponents increasing to the left.*

The work for the example of p. 83 follows:

$$
\begin{array}{lrl|ll}
\textit{Dividend:} & -15x^4+29x^3-8x^2+2x+4 & & -3x^2+4x+2 & \textit{Divisor} \\
 & \underline{-15x^4+20x^3+10x^2} & & \overline{5x^2-3x+2} & \textit{Quotient} \\
 & 9x^3-18x^2+2x+4 & & & \\
 & \underline{9x^3-12x^2-6x} & & & \\
 & -6x^2+8x+4 & & & \\
 & \underline{-6x^2+8x+4} & & & \\
 & 0 & \textit{Final Remainder} & &
\end{array}
$$

Notice the work is really the same as before; the answers also are in this case the same, but with the terms in reverse order.

But if the division is not "exact," the work and the answers change when we change the arrangement.

Thus, in the first example on p. 85:

$$
\begin{array}{lrl|ll}
\textit{Dividend:} & -24x^4-7x^3+3x^2-10x+6 & & -3x^2+4x+2 & \textit{Divisor} \\
 & \underline{-24x^4+32x^3+16x^2} & & \overline{8x^2+13x+\tfrac{65}{3}} & \textit{Quotient} \\
 & -39x^3-13x^2-10x+6 & & & \\
 & \underline{-39x^3+52x^2+26x} & & & \\
 & -65x^2-36x+6 & & & \\
 & \underline{-65x^2+\tfrac{260}{3}x+\tfrac{130}{3}} & & & \\
 & -\tfrac{368}{3}x-\tfrac{112}{3} & \textit{Final Remainder} & &
\end{array}
$$

This work is totally different from the work done before on the same example. It is for this reason that very few examples of this kind are given below. However, the student need not be surprised; for example, $31 \div 7 =$ not only $4 + \frac{3}{7}$; but also $3 + \frac{10}{7}$, and $5 - \frac{4}{7}$. Moreover, in the form on p. 85, the division may be carried to further terms, if desired, just as $31 \div 7 = 4.42 + \dfrac{.06}{7}$, in arithmetic.

If several letters occur in a problem, one of them — usually the most prominent one — is used for the purpose of arrangement. If another letter is used for arranging the expressions, the work may be wholly different, but the results will always be precisely the same if the division is exact. Thus:

Dividend :

$$
\begin{array}{llr}
2x^4 - 17x^3y + 31x^2y^2 - 23xy^3 + 12y^4 & \left|\, \underline{2x - 3y} \right. & \textit{Divisor} \\
\underline{2x^4 - \ \ 3x^3y} & \left|\, \underline{x^3 - 7x^2y + 5xy^2 - 4y^3} \right. & \textit{Quotient} \\
\quad - 14x^3y + 31x^2y^2 - 23xy^3 + 12y^4 & & \\
\quad \underline{- 14x^3y + 21x^2y^2} & & \\
\qquad\qquad 10x^2y^2 - 23xy^3 + 12y^4 & & \\
\qquad\qquad \underline{10x^2y^2 - 15xy^3} & & \\
\qquad\qquad\qquad - 8xy^3 + 12y^4 & & \\
\qquad\qquad\qquad \underline{- 8xy^3 + 12y^4} & & \\
\qquad\qquad\qquad\qquad\qquad 0 & \textit{Final Remainder} &
\end{array}
$$

Check: If $x = 1$ and $y = 1$, dividend $= 5$; divisor $= -1$; quotient $= -5$ (correct).

The student will be able to work the exercises that follow, after carefully studying these examples and trying to do them over himself without looking at the book. The examples in which the division is "not exact" are marked so that the student will see them in advance. Some of the examples are not arranged as they stand; *the student must in all cases see to it that both dividend and divisor are properly arranged before he tries to do the exercises.*

After some experience the student will be able to omit parts of the work shown above; but it is best not to do this too soon. Later (Appendix, §§ 1–4) we shall show how to reduce the labor very much.

EXERCISES X: CHAPTER IV

Perform the following divisions. In the exercises marked * the division is not exact, and the given arrangement of terms should be used; the others should be suitably rearranged, if necessary.

1. $(a^2 - b^2) \div (a - b)$.

2. $(x^2 - 3x + 2) \div (x - 1)$.

3. $(3x^2 + 5x - 2) \div (x + 2)$.

4. $(a^2 - 12ab - 13b^2) \div (a - 13b)$.

5. $(ab - ac - bd + cd) \div (a - d)$.

6. $(k^6 - 8k^3 + 16) \div (k^3 - 4)$.

7. $(a^3 - 3a^2b + 3ab^2 - b^3) \div (a - b)$.

8. $(x^4 + 4x^2y^2 + 4y^4) \div (x^2 + 2y^2)$.

9. $(x^2 - 1) \div (x - 1)$.

10. $(m^2 - n^2) \div (m + n)$.

11. $(x^3 - y^3) \div (x - y)$.

12. $(r^3 + 1) \div (r + 1)$.

13. $(m^3 + 8) \div (m + 2)$.

14. $(x^4 - 1) \div (x - 1)$.

15. $(m^4 - n^4) \div (m + n)$.

16. $(x^5 - y^5) \div (x - y)$.

17. $(A^3+3A^2B+3AB^2+B^3)\div(A^2+2AB+B^2)$.

18. $(2-x-4x^2+17x^3-12x^4)\div(2+3x-4x^2)$.

19. $(6x^4-2x^3-8x^2+4x-8)\div(3x^2-x+2)$.

20. $(6-2m-8m^2+4m^3-8m^4)\div(2-4m^2)$.

21. $(8x^5+10x^4+5x^3+15x^2-12x+2)\div(2x^2+3x-1)$.

22. $(4r^6+r^5+2r^4-4r^2-r-2)\div(4r^3-3r^2+r-2)$.

23. $(2-17r+46r^2-43r^3+20r^4-18r^5+8r^6)\div(1-4r+2r^2)$.

24. $(24y^6-28y^5+14y^4-10y^3+5y^2-1)\div(4y^2-2y-1)$.

25. $(4+4p+p^2-3p^3-2p^4+p^5+p^6)\div(4-3p^2+p^4)$.

26. $(2x^6+x^5y+4x^4y^2-2x^2y^4-xy^5-4y^6)\div(x^3+x^2y+xy^2+y^3)$.

27. $(x^5+7x^3+14x-15-11x^2-2x^4)\div(-x+3+x^2)$.

* **28.** $(x^4-x^3-7x^2+28x-11)\div(x^2-4x+7)$.

29. $(x^2-\overline{a+b}\cdot x+ab)\div(x-a)$.

* **30.** $(x^4-2x^3-10x)\div(x^2-3x+1)$.

31. $(a^4+b^4+a^2b^2)\div(a^2+b^2-ab)$.

32. $(m^5+32n^5)\div(m+2n)$.

* **33.** $x^6\div(x^2-x-2)$.

34. $(y^6-1)\div(y^2-1)$.

35. $(m^6-n^6)\div(m^3-n^3)$.

* **36.** $(k^7-4)\div(k^2-1)$.

37. $(R^3-6R^2+11R-6)\div(R^2-5R+6)$.

38. $(x^5+x+5-2x^3-7x^2+2x^4)\div(2x^2-5-x+x^3)$.

* **39.** $(r^5-4r^4+5r^3-8r^2+8r-6)\div(r^3-3r^2+2r-7)$.

55. Polynomials. The expressions in the exercises given in this chapter are all very simple. They are in fact **polynomials.** (See § 9, p. 9.)

The most general polynomial in the single letter x is

$$a+bx+cx^2+dx^3+\cdots+lx^n$$

where $a, b, c, \cdots, l$ are constant numbers, possibly zero, and where n is some positive integer.

The most general polynomial in two letters, x, y, is

$$a + bx + cy + dx^2 + exy + fy^2 + \cdots$$

(a finite number of terms), where a, b, c, $\cdots$ are constant numbers, possibly zero.

Each term of a polynomial contains each letter, if at all, only as a factor which is a simple power. If a letter occurs under a radical sign, or in the denominator of a fraction, the whole expression is *not* a polynomial in that letter.*

Thus, $3 - 2x + 5x^2$ is a *polynomial* in x, $4my^2 + 6m^2y - 7m^3$ is a *polynomial* in y and m. $5x\sqrt{y} + 3x^2y$ is a *polynomial* in x, but is *not* a *polynomial* in y (because $\sqrt{y}$ is not a simple power). The expression $2x^2 - 3 + \frac{x}{y}$ is a polynomial in x, but is *not* a polynomial in y.

All the expressions in this chapter are polynomials in all the letters in them, except some few exercises that contain fractions or contain letters as exponents. There will later be many examples that are not polynomials, in the Chapters on *Radicals* (Chapters VII, XI) and in the Chapter on *Fractions* (Chapter V). The distinction is of especial importance in *Factoring*. (See p. 91.)

REVIEW EXERCISES XI: CHAPTER IV

Perform the indicated operations; check as usual:

1. $(a^2 - a + 1)(a^2 + a + 1)$.

2. $(v - 5)(v + 3)(v - 2)$.

3. $[x^2 - (a + b)x + ab](x - c)$.

4. $(a^2 + b^2 - ab)^2$.

5. $(m^3 - 2m^2 + m - 3)(m^2 - 3m - 2)$.

6. $(2x - 3y)(3y - 5z)(5z - 2x)$.

7. $(3x^2 - 4x + 5)(2x^3 - 3x^2 - 4x - 2)(1 - 2x - 3x^2)$.

---

* Although many elementary text-books define this word so as to include any kind of terms whatever, the author of this book can find no standard authority for any other definition than that here given. See, *e.g.*, Bôcher, *Introduction to Higher Algebra*, pp. 1–4.

8. $(x^2+xy+y^2)(x^2-xy+y^2)(x^4-x^2y^2+y^4)$.

9. $(r^2+2\,rs+4\,t^2)(r^2-4\,t^2)(r^2-2\,rs+4\,t^2)$.

10. $(x^5-5\,x^4y-6\,x^3y^2+x^2y^3-2\,xy^4+y^5)(3\,x^2-2\,y^2)$.

11. $(2\,x-x^2+1)(x^3+2\,x-x^2+4)(x+3)$.

12. $(pq+q^2+p^2)(q^2+p^2-pq)(q^2-p^2)$.

13. $(8\,lmn+2\,l^3-4\,l^2m+2\,ln)(3\,ln+l^2-m)$.

14. $(ax-by)(by-cz)(cz-ax)$.

15. $(a^2x^2-abxy+b^2y^2)(b^2y^2-bcyz+c^2z^2)(c^2z^2-cazx+a^2x^2)$.

16. $\dfrac{y^5+x^5}{y+x}$. *18. $\dfrac{x^4-2\,x^2-3}{x^2+x-1}$. 20. $\dfrac{x^3+y^3-6\,xy+8}{x^2-xy+y^2-2\,x-2\,y+4}$.

17. $\dfrac{r^6-s^6}{r^2-s^2}$. *19. $\dfrac{x^3+2\,x^4-3}{x^2+1}$. 21. $\dfrac{x^3-9\,x^2+23\,x-15}{x^2-6\,x+5}$.

Show that:

22. $$\frac{(x^3-3\,x^2y+3\,xy^2-y^3)(x^2+xy+y^2)}{x^3-y^3}=(x-y)^2.$$

23. $$\frac{(9\,m^2+3\,mn+n^2)(9\,m^2-3\,mn+n^2)(9\,m^2-n^2)}{(27\,m^3-n^3)(27\,m^3+n^3)}=1.$$

24. $$\frac{(a^2+b^2+c^2)(x^2+y^2+z^2)-(ax+by+cz)^2}{(ay-bx)^2+(bz-cy)^2+(cx-az)^2}=1.$$

25. Divide $x^3-3\,x^2+5\,x-7$ by $x-a$, keeping coefficients of like powers of x together. Arrange your remainder as a polynomial in a with exponents decreasing to the right. What do you note about the remainder?

26. What would be the remainder on dividing $x^3-3\,x^2+5\,x-7$ by $x-1$? by $x-2$? by $x+1=x-(-1)$?

27. Divide $x^4-x^2+3\,x-m$ by $x-1$. What is the remainder? How may m be chosen so that the division may be exact?

28. $\dfrac{x^4-(x+1)^2}{x^2+(x+1)}$. 29. $\dfrac{1-3(1-x)+2(1-x)^2}{1-(1-x)}$.

PART III. SPECIAL MULTIPLICATIONS; FACTORS; TYPE FORMS

56. Introduction. There are some examples that occur so often in applications of algebra that it is desirable to *commit them to memory.*

As an example of what is to be done here, consider $(x + y)^2$, which means $(x + y)(x + y)$. Actually multiplying, we get,

$$\begin{array}{l} x + y \\ \underline{x + y} \\ x^2 + xy \\ \underline{\quad\quad xy + y^2} \\ x^2 + 2\,xy + y^2 \end{array}$$

Then $(x + y)^2 = x^2 + 2\,xy + y^2$. This holds whatever x and y may mean. Thus $(4 + 6)^2 = 4^2 + 2(4 \cdot 6) + 6^2 = 100$. Again $(m + n)^2 = m^2 + 2\,mn + n^2$.

To use this (or any other forms in this chapter), we try to see whether a given example can be made to *look exactly like* the known example by *pairing off the parts.*

Thus, $(3\,a + 2\,b)^2$ is exactly like $(x + y)^2$ if $x = 3\,a$ and $y = 2\,b$. Hence, since $$(x + y)^2 = x^2 + 2\,xy + y^2,$$

$$(3\,a + 2\,b)^2 = (3\,a)^2 + 2\,(3\,a)(2\,b) + (2\,b)^2 = 9\,a^2 + 12\,ab + 4\,b^2.$$

Check: If $a = 1$ and $b = 1$, $(3 + 2)^2 = 25 = 9 + 12 + 4$ (correct).

We try to remember the answer for another reason; for $(x^2 + 2\,xy + y^2) \div (x + y) = x + y$; if this example in division is given, it is convenient to know the answer.

Finally, we say the *factors* of $x^2 + 2\,xy + y^2$ are $(x + y)$ and $(x + y)$, for the product of these factors is the given expression. (See p. 9.)

The expressions, $x^2 + 2\,xy + y^2$, $x + y$, $x + y$ are all polynomials in the letters x and y. (See p. 88.) In general, in this chapter and throughout the book, we shall seek only for *polynomial* factors of *polynomials.*

This is similar to the custom in arithmetic, where we usually seek only for *integral* factors of *integers*; thus, we say the factors of 10 are 2 and 5; we do not say the factors of 10 are $\frac{5}{3}$ and 6, for example, though $\frac{5}{3} \times 6 = 10$.

Ex. 1. Find the factors of $4\,m^2 + 20\,mn + 25\,n^2$.

Given $4\,m^2 + 20\,mn + 25\,n^2$, we may possibly notice that this is the same as $(2\,m)^2 + 2\,(2\,m)(5\,n) + (5\,n)^2$. Comparing with $x^2 + 2\,x \cdot y + y^2$, we see that

$$4\,m^2 + 20\,mn + 25\,n^2 = (2\,m + 5\,n)^2.$$

The factors of $4\,m^2 + 20\,mn + 5\,n^2$ are therefore $(2\,m + 5\,n)$ and $(2\,m + 5\,n)$; in other words, $4\,m^2 + 20\,mn + 25\,n^2$ *is the square of* $(2\,m + 5\,n)$. Check by actually multiplying $(2\,m + 5\,n)$ by $(2\,m + 5\,n)$.

Ex. 2. $(21)^2 = (20 + 1)^2 = 20^2 + 2\,(20 \cdot 1) + 1^2 = 441$.

This will be found an easy way to square certain numbers.

Ex. 3. $(a + b + c)^2 = [(a + b) + c]^2 = (a + b)^2 + 2(a + b) \cdot c + c^2$
$= a^2 + b^2 + c^2 + 2\,ab + 2\,ac + 2\,bc.$

EXERCISES XII: CHAPTER IV

Perform the following multiplications and divisions; check by substitution of special values:

1. $(2\,k + 3\,c)^2$. **3.** $(y + \frac{1}{2})^2$. **5.** $(x^2 + 3)^2$. **7.** $(\frac{1}{3}\,x + \frac{1}{5}\,z^2)$.

2. $(x + 2)^2$. **4.** $(m + 3\,z)^2$. **6.** $(x^3 + 2\,x)^2$. **8.** $(x + 7.3)^2$.

9. $(a + 2\,b + 5)^2$.

10. $(2\,m + n + p)^2$.

11. $(x^2 + 3\,x + 2)^2$.

12. $(ax + by + cz)^2$.

13. $(x^2 + 24\,x + 144) \div (x + 12)$.

14. $(R^2 + 16\,R + 64) \div (R + 8)$.

15. $(p^2 + 3\,pq + 2\frac{1}{4}\,q^2) \div (p + \frac{3}{2}\,q)$.

16. $(9\,x^4 + 12\,x^3 + 4\,x^2) \div (3\,x_2 + 2\,x)$.

17. $(23)^2 = (20 + 3)^2$; 11^2; 32^2; $(4\frac{1}{2})^2 = (4 + \frac{1}{2})^2$; $(7.2)^2 = (7 + .2)^2$.

18. $(a - b)^2$. [Work this by writing $(a - b) = a + (-b)$.]

19. Of what expression do you suspect $9\,b^2 + 6\,b + 1$ to be the square? What are the factors of $9\,b^2 + 6\,b + 1$?

20. Of what expression do you suspect $x^2 + 14\,x + 49$ to be the square? What is the square of that expression?

Resolve into factors:

21. $a^4 + 2a^2b + b^2$.

22. $y^2 + 50y + 625$.

23. $r^2 + 2rs + s^2$.

24. $z^2 + 8z + 16$.

25. $p^4 + 12p^2z + 36z^2$.

26. $961 = 900 + 60 + 1$; 441; 484.

27. $a^2z^2 + 26axyz + 169x^2y^2$.

28. $(a^2 + 2ab + b^2) + 6(a + b)c + 9c^2$.

57. Square of Sum. The result of § 56 is:

$$\text{I.}\quad (x + y)^2 = x^2 + 2xy + y^2.$$

The square of the sum of two terms equals the square of the first term, plus twice the product of the two terms, plus the square of the second term.

58. Square of Difference. Likewise

$$\text{II.}\quad (x - y)^2 = x^2 - 2xy + y^2.$$

Actually multiply $(x - y)(x - y)$ to get this result.

The square of the difference of two terms equals the square of the first term, minus twice the product of the two terms, plus the square of the second term.

Ex. 1. $(4k - 3s)^2 = (4k)^2 - 2(4k)(3s) + (3s)^2$
$$= 16k^2 - 24ks + 9s^2.$$

Ex. 2. $(18)^2 = (20 - 2)^2 = (20)^2 = 2(20)(2) + 2^2$
$$= 400 - 80 + 4 = 324.$$

This will be found an easy way to square certain numbers.

Ex. 3. $(m^2 - 2mn + n^2) \div (m - n) = m - n$.

This results from the formula above.

Ex. 4.
$$\begin{aligned}(a + b - c)^2 &= [(a + b) - c]^2\\ &= (a + b)^2 - 2(a + b)\cdot c + c^2\\ &= a^2 + 2ab + b^2 - 2ac - 2bc + c^2\\ &= a^2 + b^2 + c^2 + 2ab - 2ac - 2bc.\end{aligned}$$

Ex. 5. Find the factors of $4m^2 - 20mn + 25n^2$.

As in Ex. 1, p. 92, we notice that the given expression may be written $(2m)^2 - 2(2m)(5n) + (5n)^2$. Comparing with II, we see that

$$4m^2 - 20mn + 25n^2 = (2m - 5n)^2.$$

Hence, the factors of $4m^2 - 20mn + 25n^2$ are $2m - 5n$ and $2m - 5n$. Check by actual multiplication.

EXERCISES XIII: CHAPTER IV

Perform the following multiplications and divisions; check as usual:

1. $(5r-2s)^2$. **3.** $(-a+b)^2$. **5.** $(v-2tu)^2$. **7.** $(x-2y+2)^2$.

2. $(9b-2)^2$. **4.** $(2v-13u)^2$. **6.** $(x^2-nx)^2$. **8.** $(x-y-z)^2$.

9. $(b^2-10b+25)\div(b-5)$.

10. $(16-8k+k^2)\div(4-k)$.

11. $(64x^2-16xy+y^2)\div(8x-y)$.

12. $(m^2-9m+20\frac{1}{4})\div(m-4\frac{1}{2})$.

13. $(x^4-8x^2+16)\div(x^2-4)$.

14. $[(x+y)+(x-y)]^2$.

15. $[(x+y)-(x-y)]^2$.

16. $(a-b-c+d)^2$.

17. $(a+b)^2=[a-(-b)]^2$.

18. $(19)^2=(20-1)^2$; $(29)^2$; $(9)^2$.

Resolve into factors:

19. k^2-6k+9.

20. z^4-12z^2+36.

21. $t^2-14t+49$.

22. v^2+9t^2-6vt.

23. $m^2-24mn+144n^2$.

24. $4e^2-96e+576$.

25. $64p^2q^2-80pq+25$.

26. $16x^6-40ax^3+25a^2$.

27. $9a^2x^2-42abxy+49b^2y^2$.

28. $(m^2-2mn+n^2)-2(m-n)p+p^2$.

59. Product of the Sum and Difference. Likewise,

$$\text{III.}\quad (x+y)(x-y)=x^2-y^2.$$

Actually multiply $(x+y)$ by $(x-y)$ to get this result.

The product of the sum and the difference of the same two terms equals the square of the first term, minus the square of the second term.

Ex. 1. $(3a+2b)(3a-2b)=(3a)^2-(2b)^2=9a^2-4b^2$.

Ex. 2. $(a+b+c)(a+b-c)=[(a+b)+c][(a+b)-c]$

$$=(a+b)^2-c^2=a^2+2ab+b^2-c^2.$$

Check by actually multiplying.

Ex. 3. Find the factors of $4a^2-b^2$. We write

$$4a^2-b^2=(2a)^2-b^2=(2a+b)(2a-b).$$

Hence, $2a+b$ and $2a-b$ are the factors.

EXERCISES XIV: CHAPTER IV

Perform the following multiplications and divisions; check as usual:

1. $(4a+b)(4a-b)$.
2. $(2x+3y)(2x-3y)$.
3. $(x^2+5)(x^2-5)$.
4. $(x^2+5)(-x^2+5)$.
5. $(m-10kz)(m+10kz)$.
6. $(a-b-c)(a+b-c)$.
7. $(a-b-c)(a-b+c)$.
8. $(a-b-c)(a+b+c)$.
9. $(a+2b-3c)(a+2b+3c)$.
10. $(z^2-16r^2)\div(z-4r)$.
11. $(a^4-b^6)\div(a^2+b^3)$.
12. $(m^2n^4-m^4n^2)\div(mn^2+m^2n)$.
13. $(25n^2-49m^2)\div(5n-7m)$.
14. $(64p^2q^2-25)\div(8pq+5)$.
15. $99\div 11=(100-1)\div(10+1)$.
16. $99\div 9$.
17. $624\div 26$.
18. $24\cdot 26=(25-1)(25+1)$.
19. $15\cdot 17$.
20. $35\cdot 37$.
21. $(a^2+2ab+b^2-16c^2)\div(a+b-4c)$.
22. $(a^2+6ab+9b^2-4x^2+4xy-y^2)\div(a+3b-2x+y)$.

Resolve into factors:

23. $100r^2-169s^2$.
24. $4A^2-9a^2$.
25. x^4-64x^2.
26. $a^2-2ax+x^2-4y^2$.
27. $a^2-2ab+b^2-9c^2$.
28. x^6-x^2.
29. $y^4-b^2+2bc-c^2$.
30. $16-x^4y^4$.
31. $z^4+z^2+1=(z^4+2z^2+1)-z^2$.
32. $a^2+2ab+b^2-c^2+6cy-9y^2$.
33. $a^2-2ab+b^2-4l^2+12lm-9m^2$.

60. Likewise,

$$\text{IV.}\quad (x+a)(x+b)=x^2+(a+b)x+ab.$$

Actually multiply $(x+a)$ by $(x+b)$ to get this result.

The product of two binomials whose first term is the same equals the square of the common term, plus the sum of the second terms times the common term, plus the product of the second terms.

Ex. 1. $(x+2)(x+4)=x^2+6x+8.$

Ex. 2. $(x-3)(x+5)=[x+(-3)][x+5]$
$$=x^2+[(-3)+5]x+(-3)(5)$$
$$=x^2+2x-15.$$

Notice that the rule really applies to differences as well as to sums, as in example 2. We need only express the difference $x-3$ as the *sum* of x and -3.

On the same principle $(x-y)^2$ can be worked out by the rule for $(x+y)^2$. Thus,

$$(x-y)^2=[x+(-y)]^2=x^2+2x(-y)+(-y)^2=x^2-2xy+y^2,$$

which is the same as the result in § 52. (See Ex. 18, List XII.)

This rule is used chiefly *to find the factors of given expressions like the products above.*

Ex. 3. Find the factors of x^2+6x+8.

We write $x^2+6x+8=(x+?)(x+?)$.

Now the last terms have a product 8. Hence, we try such combinations as 1 and 8, and 2 and 4, etc. The pair 1 and 8 is not correct, for $(x+1)(x+8)=x^2+9x+8$, which is *not* the given expression. In fact, it is clear that the *sum* of the numbers of the correct pair must be 6. We want, then, a pair of numbers whose *product* is 8 and whose *sum* is 6. If we try a few pairs, we shall probably try 2 and 4 quite soon; this pair is correct, for $2\times 4=8$ and $2+4=6$. Checking the answer by multiplication, we find $(x+2)(x+4)=x^2+6x+8$. Hence, the required factors are $x+2$ and $x+4$.

If, after trying all pairs whose product is 8, we find no pair that is correct, we must give up the problem just now; later we shall solve such problems by a different method.

Ex. 4. Consider $x^2+2x-15$, the result of example 2.

We must choose a pair of numbers whose *product* is -15. Such pairs are -1 and $+15$, $+1$ and -15, $+3$ and -5, -3 and $+5$. But the *sum* of the pair must be 2; hence the last pair is the correct one since $-3+5=2$.

Check: $(x-3)(x+5)=x^2+2x-15$. The required factors are therefore $(x-3)$ and $(x+5)$.

Ex. 5. Similarly, $1+2x-15x^2=(1-3x)(1+5x)$,

and $x^2+2xy-15y^2=(x-3y)(x+5y)$.

The letters used should not confuse the student.

EXERCISES XV: CHAPTER IV

Perform the following multiplications:

1. $(q+2)(q-3)$.

2. $(z+1)(z+4)$.

3. $(m-2n)(m+n)$.

4. $(r-s)(r-5s)$.

5. $(x-3)(x+5)$.

6. $(a+6)(a+2)$.

7. $(1-s)(1+3s)$.

8. $(l+2m)(l+7m)$.

9. $(v^2-2)(v^2-9)$.

10. $(x^3+3)(x^3-4)$.

11. $(pq-8)(pq+1)$.

12. $(t-3a)(t-9a)$.

13. $(xy^2-2)(xy^2+10)$.

14. $(s+t-1)(s+t-3)$.

15. $(xyz+2t)(xyz+4t)$.

16. $(1-2c^3)(1+3c^3)$.

17. $(x-\frac{1}{2})(x-\frac{3}{2})$.

18. $(b-\frac{3}{5}a)(b+\frac{8}{5}a)$.

19. $(2z-9)(2z+8)$.

20. $(6pq-2rs)(6pq-3rs)$.

Resolve into factors:

21. $z^2+az-6a^2$.

22. a^2+6a+8.

23. a^2-6a+8.

24. t^2-2t-3.

25. x^2-5x+6.

26. x^2+5x+6.

27. x^2-6x+5.

28. x^2+6x+5.

29. $1-3z-18z^2$.

30. $m^2-2mn-15n^2$.

31. $r^2-12rs+35s^2$.

32. $1-12rs+35r^2s^2$.

33. $y^2-14xy+24x^2$.

34. $m^4+5m^3n-24m^2n^2$.

35. $t^2-15tu+56u^2$.

36. $4u^2-12u+5=(2u)^2-6(2u)+5$.

37. $9x^2+6x-8$.

38. $25+15x-4x^2$.

39. $4-8pq-21p^2q^2$.

40. $a^2t^2-11at-26$.

41. $c^4d^2-13c^2de^2+40e^4$.

42. $16x^2-32xz+15z^2$.

43. $x^2y^2-15xyz-34z^2$.

44. $x^2-(z+2)x+2z$.

45. $x^2y^2z^2-(3-t)xyz-3t$.

46. $a^2-(x-y)a-xy$.

47. $f^4+7f^2r-30r^2$.

48. $25-10q-48q^2$.

49. $x^2-2xy+y^2-9(x-y)+20=(x-y)^2-9(x-y)+20$.

50. $p^2+6pq+9q^2+5(p+3q)-14$
$=(p+3q)^2+5(p+3q)-14$.

61. Form V: ax^2+bx+c. The product $(2x+3)(4x+5)$ $=8x^2+22x+15$, as is seen by multiplying $(2x+3)$ and $(4x+5)$. If we did not know the factors of $8x^2+22x+15$, we could find them as follows: We know $8x^2+22x+15$ $=(?\,x+\,?)(?\,x+\,?)$. Now the product of the coefficients of x must be 8, and the product of the last two terms must be 15. We therefore try all factors that meet these conditions:

$$\begin{array}{c} 2x+5 \\ 4x+3 \\ \hline 8x^2+26x+15 \\ \text{(wrong)} \end{array} \qquad \begin{array}{c} 8x+15 \\ x+\ \ 1 \\ \hline 8x^2+23x+15 \\ \text{(wrong)} \end{array}$$

$$\begin{array}{c} 2x-3 \\ 4x-5 \\ \hline 8x^2-22x+15 \\ \text{(wrong)} \end{array} \qquad \begin{array}{c} 2x+3 \\ 4x+5 \\ \hline 8x^2+22x+15 \\ \text{(correct)} \end{array}$$

This may take a long time, for there are usually many possibilities that occur to one before the correct combination is thought of. If we notice that the *middle term* is the one that comes out wrong in case of a wrong guess, we need try only it, assuming that the others will be right. We notice that the middle term is made as follows: the *cross products* are those marked below; their sum is the middle term:

$$\begin{array}{cc} \multicolumn{2}{c}{2x+5} \\ \multicolumn{2}{c}{4x+3} \\ \hline 20x & 6x \end{array} \qquad \begin{array}{cc} \multicolumn{2}{c}{2x-3} \\ \multicolumn{2}{c}{4x-5} \\ \hline -12x & -10x \end{array} \qquad \begin{array}{cc} \multicolumn{2}{c}{2x+3} \\ \multicolumn{2}{c}{4x+5} \\ \hline 12x & 10x \end{array}$$

We therefore try just these cross products, and decide that our choice was wrong if we do not get the correct middle term.

Ex. 1. Factor $4x^2-13x+10$.

Trying various pairs of numbers whose product is 4 with other pairs whose product is 10, we finally try $(4x-5)(x-2)$; the cross products are $(-2)(4x)=-8x$ and $(-5)(x)=-5x$; their sum is $(-8x)+(-5x)=-13x$, which is the correct middle term. We therefore multiply $(4x-5)$ by $(x-2)$ to test our result. This gives $4x^2-13x+10$; hence, this pair is correct and the required factors are $(4x-5)$ and $(x-2)$.

Ex. 2. $8 + 22x + 15x^2 = (2 + 3x)(4 + 5x)$.

Ex. 3. $8m^2 + 22mn + 15n^2 = (2m + 3n)(4m + 5n)$.

The letters used should not confuse the student.

EXERCISES XVI: CHAPTER IV

Perform the following multiplications:

1. $(2a - 3)(a - 4)$.
2. $(3a + 1)(5a - 3)$.
3. $(z - 6k)(3z + k)$.
4. $(5 - x)(7 + 2x)$.
5. $(x + 5y)(5x - y)$.
6. $(3x - 2az)(4x + 3az)$.

Resolve into factors:

7. $8z^2 + 8z - 6$.
8. $3x^2 + xz - 52z^2$.
9. $2 - a - 21a^2$.
10. $6p^2q^2 - 7pq - 20$.
11. $15 - r - 2r^2$.
12. $2r^2 + r - 15$.
13. $7x^2 + xy - 6y^2$.
14. $6m^2 - 13mn + 6n^2$.
15. $6\left(\frac{m}{n}\right)^2 - \frac{m}{n} - 12$.
16. $2x^4 - x^2 - 45$.
17. $12x^2y^2 + 17xyz + 6z^2$.
18. $12g^2 + gh - 6h^2$.
19. $2ax^2 + (4 - 3a)x - 6$.
20. $abx^2 + (a + b)tx + t^2$.
21. $12x^2 - (4b - 6)x - 2b$.
22. $4e^2 + 960e + 479$.

62. Other Forms. There are many other general results which might be given here. Among them we mention the following:

(1) $(a + b)(a^2 - ab + b^2) = a^3 + b^3$. See Ex. 10, p. 82.

(2) $(a - b)(a^2 + ab + b^2) = a^3 - b^3$. See Ex. 13, p. 82.

(3) $(a^2 - b^2)(a^2 + b^2) = a^4 - b^4$, or

(3 a) $(a - b)(a + b)(a^2 + b^2) = a^4 - b^4$.

[This is really an application of § 59.]

(4) $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$. See Ex. 35, p. 83.

(5) $(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.

See Ex. 36, p. 83.

(6) $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$. See Ex. 38, p. 83.

(7) $(a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$.
See Ex. 39, p. 83.

These and many others may be found by the student by multiplying together the given factors.

As before, if an expression can be put into a form exactly similar to one of these answers, its factors may be found by comparison with the factors given above on the left.

EXERCISES XVII: CHAPTER IV

Factor:

1. $p^3 - 8$. **3.** $x^4 - 81$. **5.** $.16m^4 - n^8$. **7.** $p^6 - q^6$.

2. $p^3 + 8$. **4.** $27 - 125a^3$. **6.** $r^3 - t^6$. **8.** $a^8 - b^8$.

9. $x^3 + 3x^2 + 3x + 1$.

10. $t^3 - 6t^2 + 12t - 8$.

11. $1 + 4t + 6t^2 + 4t^3 + t^4$.

12. $m^3n^3 + 216x^3$.

13. $(a+b)^3 + (a-b)^3$.

14. $x^3 - 30x^2 + 300x - 1000$.

15. $p^4 - \frac{4}{3}p^3 + \frac{2}{3}p^2 - \frac{4}{27}p + \frac{1}{81}$.

16. $x^3 + 3x^2y + 3xy^2 + y^3 - z^3$.

17. $a^3 - 3a^2b + 3ab^2 - b^3 - 8c^3$.

18. $x^3 - 3x^2 + 3x - 1 - 125t^6$.

Perform, by factoring, the operations indicated:

19. $\dfrac{a^3 + 3a^2b + 3ab^2 + b^3}{a^2 + 2ab + b^2}$. **20.** $\dfrac{k^4 - 4k^3 + 6k^2 - 4k + 1}{k^3 - 3k^2 + 3k - 1}$.

Perform the following multiplications, and express the result as a formula in each case:

21. $(a^2+ab+b^2)(a^2-ab+b^2)$. **22.** $(a^3 - b^3)(a^3 + b^3)$.

23. $(a+b)^5$. **24.** $(a-b)^5$. **25.** $(a \pm b)^6$.

63. Factoring by Grouping. Very often expressions may be factored by the simple process of **grouping the terms** together. This method, which is one of the most helpful methods known, is based directly on the laws of p. 35. Rule V, p. 35:

$$a(b+c) = ab + ac$$

is in itself an example of factoring, and is the chief rule used here; this has already been used extensively in multiplication and division. The same method has been used on p. 79, § 51, in finding monomial factors. In addition to this, expressions, after rearrangement, may be compared with the type forms given above.

Ex. 1.
$$\begin{aligned} & ax + by + bx + ay \\ &= ax + bx + ay + by \\ &= x(a+b) + y(a+b) = (x+y)(a+b). \end{aligned}$$

Hence, the factors of $ax + by + bx + ay$ are $(x+y)$ and $(a+b)$.

Check: by actually multiplying $(x+y)$ and $(a+b)$.

Ex. 2.
$$\begin{aligned} & a^2 + 4\,ab + 4\,b^2 - 9\,x^2 \\ &= (a^2 + 4\,ab + 4\,b^2) - 9\,x^2 \\ &= (a + 2\,b^2)^2 - (3\,x)^2 \\ &= [(a + 2\,b) + 3\,x][(a + 2\,b) - 3\,x]. \end{aligned}$$

Examples similar to this one have already been given in the preceding lists.

EXERCISES XVIII: CHAPTER IV

Factor the following expressions:

1. $ab - a^2 - b^2 + ab.$

2. $xy + 6 + 2\,x + 3\,y.$

3. $a^3 - 1 - a + a^2.$

4. $ab + ac - bc - b^2.$

5. $ap + 3\,qx + 3\,aq + px.$

6. $3 + 2\,z - 6\,c - 4\,cz.$

7. $ab^2 - abc - bx + cx.$

8. $x^3 - 4\,x^2 + 3\,x - 12.$

9. $a^2 - ap - aq + pq.$

10. $x^3 - ax^2 + bcx - abc.$

11. $rst + s^2t^2 - 2\,r^2s - 2\,rs^2t.$

12. $8\,z^3 - 6\,zt + 15\,t^2 - 20\,z^2t.$

13. $t^3 - yt^2 - x^2t + x^2y.$

14. $t^3 + at^2 + ct^2 + bt + act + ab.$

15. $x^{m+n} - ax^m - bx^n + ab.$

16. $1 - y - z + yz.$

17. $x^2 - a^2 [= x^2 + ax - ax - a^2].$

18. $x^2 + 2\,ax + a^2 [= x^2 + ax + ax + a^2].$

19. $x^2 - 2\,ax + a^2.$

20. $x^3 - a^3.$

REVIEW EXERCISES XIX: CHAPTER IV

Factor:

1. $x^2+6x-16$.
2. $x^{12}-4$.
3. $p^2+2pq+q^2-3(p+q)-10$.
4. z^3-64.
5. $4a^2-20ax+25x^2$.
6. $a^3x^3-3a^2b^2x^2+3ab^4x-b^6$.
7. x^4-16.
8. $p^2-10pq+25$.
9. $10-cz-60c^2z^2$. ✓
10. $x^4-12x^3+54x^2-108x+81$.
11. $27x^6+8y^3$.
12. $a^2+2ab+b^2-c^6$.
13. $3z^4-8z^2-35$.
14. $9+18a-7a^2$. ✓
15. $1-v-156v^2$.
16. $81L^4-256$.
17. $10-x-3x^2$.
18. p^4-81q^4.
19. $63=64-1$; $1001=1000+1$; $65=64+1$; $77=81-4$.
20. $m^4-4m^3n+6m^2n^2-4mn^3+n^4-16p^4$.
21. $k^4-12k^3+54k^2-108k-175=(k-3)^4-256$. [Factor.]
22. $a^6-b^6=(a^3-b^3)(a^3+b^3)$. [Factor these factors.]
23. $a^4+a^2b^2+b^4=(a^4+2a^2b^2+b^4)-a^2b^2$. [Factor.]
24. $a^6-b^6=(a^2-b^2)(a^4+a^2b^2+b^4)$. [Factor these factors.]
25. $a^8-b^8=(a^4-b^4)(a^4+b^4)$. [Factor these factors.]
26. $a^{10}-b^{10}=(a^5-b^5)(a^5+b^5)$. [Factor these factors.]
27. $a^{10}-b^{10}=(a^2-b^2)(a^8+a^6b^2+a^4b^4+a^2b^6+b^8)$. [Factor.]
28. $a^8+a^6b^2+a^4b^4+a^2b^6+b^8$. [Make use of Exs. 26, 27.]
29. $a^2-2ab+b^2-x^2+2xy-y^2$.
30. $a^3-3a^2b+3ab^2-b^3-x^3+3x^2y-3xy^2+y^3$.
31. $Ax^2-3Ax+2A+Bx^2-3Bx+2B$.
32. $p^2r+2p^2t+7rp+14pt+24t+12r$.
33. $1-r-s-t+rt+rs+st-rst$.

PART IV. APPLICATIONS. ENGLISH TRANSLATED INTO ALGEBRA

64. Expression of English in Formulas. The preceding chapters have given practice in writing formulas in the place of English.

Thus, instead of the rule: "The square of the sum of two terms is equal to the square of the first term, plus twice the product of the two terms, plus the square of the second term," we merely write

$$(x + y)^2 = x^2 + 2\,xy + y^2,$$

which really says the same thing in much shorter form.

Again, instead of saying, "The volume of a box in cubic inches is the product of its width times its length times its height, each measured in inches," we may simply say

$$v = w \cdot l \cdot h,$$

where v means volume in cubic inches, and w, l, h, stand for the width, length, and height, respectively, measured in inches.

In many cases the student will find it easier to write down the formulas if he first tries the problem with known numerical values. In doing so it is advisable not to perform the additions, multiplications, etc., but merely to indicate them as a guide in the case when no *numbers* are actually given. In this way the **structure** of the example is seen with simple numbers, and can be followed afterwards in the use of letters for unknown numbers.

Consider the problem:

Ex. 1. A tin box is to be made from a square piece of tin by cutting square pieces out of the corners and then folding up the flaps. Find the size of the piece of tin which must be used to make a box 4 in. high that shall contain 100 cu. in. volume.

Let us first become familiar with the problem by trying several numbers. Suppose the original plate were 18 in. square (in fig-

ure, $AB = BC = 18$ in.), and we cut out the shaded corners, each 4 in. square. Then $HE = 18$ in. $- 2 \times 4$ in. $= 10$ in. Likewise $FE = 10$ in. If we now fold up the flaps, we get a box whose bottom is 10 in. square and whose height is 4 in. The volume of this box is 10 in. $\times$ 10 in. $\times$ 4 in. $= 400$ cu. in.

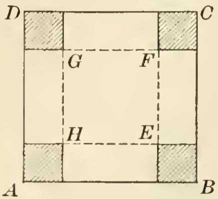


Fig. 18.

It is now easier to try the given problem. To start with, we do not know how long AB must be. Let us call its length l in. and then try to find l. If we cut the corners out as before, we shall have $HE = l$ in. $- 2 \times 4$ in. $= (l - 8)$ in. just as before. Likewise $FE = (l - 8)$ in. Hence, our box will have its bottom $(l - 8)$ in. square. Its height will be 4 in. Its volume is then $(l - 8)$ in. $\times (l - 8)$ in. $\times$ 4 in. $= 4(l - 8)^2$ cu. in. If this volume is to be 100 cu. in., we shall have

$$4(l - 8)^2 = 100.$$

Divide both sides by 4: $(l - 8)^2 = 25,$

hence, $(l - 8) = 5$ or else $- 5,$

since $(5)^2 = 25$, and also $(-5)^2 = 25.$*

Adding 8 to each side, we get $l = 8 + 5$ or else $8 - 5.$

It follows that $l = 13$ or else $l = 3.$

The correct result is $l = 13$, for $l = 3$ would not really do at all; we could not cut corners 4 in. square out of a piece of tin 3 in. square.

As in this problem the student must always be careful to see which, *if any*, of several possible answers are the correct ones, for there may be answers that cannot possibly mean anything, as is seen above.

Ex. 2. If a printed book is to have a margin 2 in. wide, how large must the pages be cut if there is to be 70 sq. in. of print on them and the pages are to be twice as long as they are wide?

---

* There are no other numbers whose square is 25, for a positive number less than 5 would give a square that is too small; a positive number greater than 5 would give a square that is too large; and similarly no negative number except -5 would produce precisely 25. It is very important, in all problems, to make sure that the answers found are all that exist. If this is not done, some answer — perhaps the most important — may be overlooked.

Let us first become familiar with the problem. Suppose the page is 10 in. $\times$ 20 in. (twice as long as wide).

Then $AB = 10$ in., $BC = 2 \times 10$ in. $= 20$ in.

If the margin (on each edge) is 2 in. wide, then

$$EF = 10 \text{ in.} - 2 \times 2 \text{ in.} = 6 \text{ in.},$$

and $$FG = 20 \text{ in.} - 2 \times 2 \text{ in.} = 16 \text{ in.}$$

The printed portion is then (6×16) sq. in. $= 96$ sq. in. Suppose now that the width AB is not known in advance; call it w in. Then

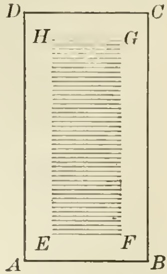


FIG. 19.

$$AB = w \text{ in.} \quad BC = 2 \times w \text{ in.} = 2\,w \text{ in.}$$

If the margin (on each edge) is 2 in. wide, then

$$EF = (w - 2) \times 2 \text{ in.} = (w - 4) \text{ in.},$$

and $$FG = [2\,w - 2 \times 2] \text{ in.} = (2\,w - 4) \text{ in.}$$

Hence, the printed portion is

$$(w - 4)(2\,w - 4) \text{ sq. in., or } (2\,w^2 - 12\,w + 16) \text{ sq. in.}$$

But the printed portion is to be 70 sq. in., hence,

$$2\,w^2 - 12\,w + 16 = 70.$$

Divide each side by 2:

$$w^2 - 6\,w + 8 = 35.$$

Subtract 35 from each side:

$$w^2 - 6\,w - 27 = 0.$$

But this is $$(w + 3)(w - 9) = 0 \quad (\S\ 60).$$

Hence, $$w + 3 = 0, \text{ or else } w - 9 = 0,$$

for the product $(w + 3)(w - 9)$ would not be zero unless one of the factors were zero.

If $w + 3 = 0$, then $w = -3$; this is meaningless, for a piece of paper cannot be -3 in. wide.

If $w - 9 = 0$, then $w = 9$; and this is soon seen to be correct, for if $w = 9$, the actual printed area is 5 in. wide and 14 in. long; its area is therefore (5×14) sq. in. $= 70$ sq. in., as was required.

65. Simple Changes in Equations. We have made several changes in the equations above which we shall now review.

Thus, we had in example 1, $4(l - 8)^2 = 100$ and we *divided each side by* 4. The justification for this is that $4(l - 8)^2$ *is the same number* as 100; hence, $\frac{1}{4}$ of either of them is equal to $\frac{1}{4}$ of the other. (Compare § 35, p. 55.)

Likewise, we multiplied both sides by the same number, added the same number to both sides, subtracted the same number from both sides. The argument is the same.

We see that in any equation, if it is a true equation, *the two sides stand for the same number;* hence:

Given any equation, we may

I. *Add the same number to each side.*

II. *Subtract the same number from each side.*

III. *Multiply each side by the same number.**

IV. *Divide each side by the same number except zero.*†

V. *Perform the same operation (of any kind) on the whole of each side, if the result is known to be a single number in each case.*‡ (Take care to perform the operation on the *whole* of each side, not on a part of it.)

These principles have been used frequently; the first four were stated in another form in § 35, p. 55.

66. Product Equal to Zero. Another principle is the one used in example 2, p. 104. Since $(w+3)(w-9)=0$, we knew that either $(w+3)=0$, or that $(w-9)=0$, for otherwise the product of the two could not be zero.

---

* If both sides are multiplied by zero, the new equation is $0=0$, which is correct, but not useful. Avoid multiplying both sides by zero, and test any expression used as a multiplier to see if it is zero for the values finally found.

† In dividing, great care is sometimes necessary to avoid dividing by zero. Notice that no number can be divided by zero. (See p. 75.)

‡ If the result of the operation performed is not a single number, great care is necessary. Thus, we said, in example 1, if $(l-8)^2=25$, we know that $l-8$ is *either* $+5$ or -5. To get this we *take the square root of each side*. But 25 has *two* square roots, since $(+5)^2=(+5)\times(+5)=25$ and $(-5)^2=(-5)\times(-5)=25$. Hence, we cannot be sure which one of these is equal to $(l-8)$. Nevertheless, we may make a perfectly definite statement even in this case; namely, "$(l-8)$ is equal *either* to 5 *or else* to -5."

This principle is simply that two numbers, neither of which is zero, have a product which is not zero. This truth is quite evident after a little thought, and we shall state it as follows:

VI. *If the product of two factors is zero, then at least one of the factors is zero.*

Let us solve additional problems to illustrate these principles:

Ex. 1. Each member of a certain family gave each of the others a present, at Christmas, which cost 50 cents. If the total spent by the family was \$10, how many persons are there in the family?

To become familiar with the problem, suppose there had been 6 members. Then each one would have given 5 presents, so that all together 6×5, or 30, presents would have been given.

Since the number in the family is not really known, let us call it n. Then each member gave $n - 1$ presents, so that all together there were $n(n-1)$ presents given. If each cost 50 cents, the cost in cents was $50\,n(n-1)$. But this is known to be 1000 cents:

$$50\,n(n-1) = 1000.$$

Divide both sides by 50:

$$n(n-1) = 20.$$

Multiply $n - 1$ by n, and subtract 20 from each side,

$$n^2 - n - 20 = 0,$$

$$(n+4)(n-5) = 0.$$

Hence, one or the other of these factors is zero:

$$n + 4 = 0, \text{ or } n - 5 = 0;$$

that is, subtracting 4 from each side *or* adding 5 to each side gives,

$$n = -4, \text{ or } n = 5.$$

Now $n = -4$ is meaningless in this problem; therefore $n = 5$: there are 5 persons in the family. To check this result, we notice that there would then be $5 \times 4 = 20$ presents, which, at 50 cents each, would cost \$10. This check is complete.

Many practical problems arise in computing the effect of errors in measurements. In such problems the error

in measurement is usually counted positive when the measurement is too large; if the measurement is too small, the error is said to be negative. The following examples will illustrate the calculations of such error effects.

Ex. 2. Let e denote the error (in inches) made in the measurement of the side of a square whose side is really 10 ft. long; and let E denote the error (in square inches) in the computed value of the area of the square. Express E in terms of e. If $e = 3$ (in.), find E. If $E = 484$ (sq. in.), find e. If $E = -119\frac{3}{4}$ (sq. in.) (*i.e.* the computed area is $119\frac{3}{4}$ sq. in. too small), find e.

The side is really 10 ft., or 120 in. long. If the error is e, the measurement is $120 + e$ (in inches). Hence, the computed area is $(120 + e)^2$ (in square inches). Since the real area is $(120)^2$ (in square inches), the error E in the computed area is

$$E = (120 + e)^2 - (120)^2 = e^2 + 240\,e.$$

If $e = 3$ (in inches), the value of E is found by putting 3 in place of e in the preceding equation; and we find

$$E = 3^2 + (240)(3) = 9 + 720 = 729;$$

hence, the effect of an error of 3 in. in measuring the length of the side of the square causes an error of 729 sq. in. in the computed area, or about 5 sq. ft.

If $E = 484$ (in square inches), we have

$$e^2 + 240\,e = 484, \quad \text{or} \quad e^2 + 240\,e - 484 = 0;$$

whence, $$(e - 2)(e + 242) = 0;$$

and $$\text{either } e - 2 = 0, \quad \text{or} \quad e + 242 = 0;$$

that is, $$\text{either } e = 2, \quad \text{or} \quad e = -242.$$

The answer $e = -242$ is unreasonably large, since no one would conceivably make an error of the amount of 242 in. in measuring a length of 10 ft. The answer $e = 2$ is evidently the only one to which we need pay attention; it means that the error of 484 sq. in. in the computed area would be caused by an error of 2 in. in the measurement of the side, the measurement being 2 in. too long, since e is positive.

If $E = -119\frac{3}{4}$ (in square inches), we have

$$e^2 + 240\,e = -119\tfrac{3}{4}, \quad \text{or} \quad e^2 + 240\,e + 119^3 = 0,$$

or, multiplying by 4, we get

$$4\,e^2 + 960\,e + 479 = 0.$$

Since 479 has no factors except itself and 1, we would soon try the factors $(2e + 1)$ and $(2e + 479)$; these are correct, as will be seen by multiplying them together. Since their product is zero, we have

$$\text{either } 2e + 1 = 0, \text{ or } 2e + 479 = 0,$$

that is,

$$\text{either } e = -\tfrac{1}{2}, \text{ or } e = -239\tfrac{1}{2}.$$

The answer $e = -\frac{1}{2}$ means that the error made in measuring the side was *negative*, *i.e.* that the measurement was too short, and the amount of the error was $\frac{1}{2}$ in. The other answer $e = -239\frac{1}{2}$ is unreasonably large, and we need not consider it.

Check: If $e = -\frac{1}{2}$, the measured length was $119\frac{1}{2}$ in.; hence the computed area was $(119\frac{1}{2})^2$ (sq. in.) or $14{,}280\frac{1}{4}$ sq. in. The real area being 14,400 sq. in., the error E was $-119\frac{3}{4}$ sq. in. (correct).

In the preceding problem, the answer $e = -\frac{1}{2}$ has a real meaning, though it is negative. In problems about actual things, always notice carefully whether or not a negative answer can be interpreted. Notice also whenever an answer is unreasonably large, or when it is unreasonable for any other cause, and be careful to say *why* any of several possible answers is discarded. See also the footnote on p. 104.

67. Solution of Problems. The following exercises will illustrate the uses of factoring and of multiplication and division in simple cases. They may all be solved by the principles of §§ 65–66; and the answers found will be correct if these principles are carefully followed.

The equations solved above are **quadratic** equations; we shall study such equations in more detail in Chapter VIII, p. 203.

Apparent answers may be found which do not fulfill the conditions of the given problem. Thus, the equation (1) $x - 3 = 0$ has only one answer, $x = 3$. But if both sides of equation (1) are multiplied by $x - 2$, the resulting equation $(x - 2)(x - 3) = 0$ has two possible answers, $x = 2$ and $x = 3$, the first of which, $x = 2$, is not a possible answer for equation (1). No such false answer can arise if both sides are multiplied by a *number*, as required in III, § 65, if that number is not zero.

In order to be sure that no mistakes have been made in the work, as well as to avoid the possible introduction of false answers, each answer should be checked, as above, by actually trying it in the given problem. Such a check is complete.

EXERCISES XX: CHAPTER IV

1. In example 1, p. 103, we found that the given problem led to the formula $v = 4(l-8)^2$. Find v when $l = 8$; when $l = 9$; 10; 11; 13. Find v when $l = 0$; 1; 3; 5; 7; -1; -5. Make a table of these values of v and l, and plot the graph for this equation. Does the shape of the graph resemble any you have drawn before?

When $v = 100$, determine l from the graph. This procedure is called the *graphical solution* of the problem. Compare your results with the results on p. 104.

2. When $v = 72\frac{1}{4}$, $l = ?$ What is v when $l = 2$? when $l = 3\frac{1}{2}$? Solve both graphically and algebraically.

For any value of l, how many values of v are there? For any value of v, how many values of l are there? In the case of two results such as $v = 100$, $l = 13$ or 3, how is the correct answer to be distinguished from the incorrect one in the figure?

3. In example 2, p. 104, the printed portion $p = 2w^2 - 12w + 16$, if $w =$ width. Find p if $w = 0$; 1; 2; 5; 10; -1; -5; -10. Make a table and draw the graph. Solve graphically for w if $p = 70$, and compare your results with those on p. 105. How many values of w are there? Point out why one is impossible as a true solution.

4. A certain rectangular yard is known to contain 650 square feet; it is found that a 150-foot rope exactly surrounds the fence. What are the length and breadth of the yard? Solve by letting $l =$ length.

5. If A denotes the area of a field, as in Ex. 4, what relation connects A and l? Draw the graph. Solve graphically for l if $A = 650$. What can you say of the two values obtained for l in this case?

6. For what values of l in Ex. 5 is $A = 0$? What is the space between these values? If the number l represents the *length* of the field, what distance in the figure represents

the breadth of the field? Point out on the figure the greatest possible value of the area A. For this value of A, what are the length and breadth of the field?

7. By what amount must the length and breadth of a rectangular plot of ground 300 ft. long by 50 ft. wide be equally increased, in order that the area may be increased by 3600 sq. ft.?

8. Represent by a graph the increase in area of the plot of ground corresponding to equal increase of length and breadth. Solve the problem graphically. Interpret negative values. What equal decrease of length and breadth will reduce the area 5025 sq. ft.?

9. What error (e) in measuring the side would cause an error (E) of 241 sq. in., the computed area of the square mentioned in Example 2, § 66? If $e = 5$, find E; if $e = -3$, find E; if $E = -239$, find e.

10. The sum of the squares of two consecutive integers is 113. What are the integers?

Have both positive and negative results a meaning in this problem?

11. "Think of a number; add 3 to it; multiply the result by 2; from this result subtract 5, then multiply by the number you first thought of. What is your result?" If the result given is 36, what is the number thought of? If the result given is 21, what is the number thought of?

12. Show that the difference of the squares of two consecutive integers n and $n + 1$ must be an odd number. If the difference of the squares is given, how may the numbers be found?

Determine two consecutive integers such that the difference of their squares is 9, 15, 33.

13. For the frame of a picture 10 by 15 in., 116 sq. in. of material are available. How wide may the frame be made if the frame is the same width on all sides, and if it exactly meets the picture?

Solve graphically, also.

14. The interest on a sum, p, of money at a given rate, r, for a given time, t, is $i = prt$. What is the amount, A?

What is the amount, A, for one year? What is the amount, A', at annually compounded interest, for two years?

Plot the annually compounded interest for two years on \$200 at various rates. (Do not start the plotting of A' from 0, but (say) from \$200. Choose rates from 0% to 12% at intervals of about $\frac{1}{2}$%.) For what rate will A' be \$220.50? Solve first graphically, then by the equation.

15. A rectangular frame is so constructed that by means of a slide its width may be altered. When it is set at the width 5 centimeters, a rod 13 centimeters long just fits as diagonal. How much must the width be increased so that a rod 15 centimeters long may fit as diagonal?

16. A rectangular plot of ground 60 feet by 14 feet is to be doubled in area by equal increase of length and breadth. Draw a picture illustrating in general the alteration in area corresponding to equal alteration of length and breadth. Find the alteration necessary, both by use of the graph and by solution of the equation.

17. Let the error in the measurement of the side of a square whose side is really 15 ft. long be denoted by e (in inches), and the error in the computed area by E (in square inches). Express E in terms of e. If $e = 1$, find E; if $e = 2$, find E; if $e = 3$, find E; etc.; if $e = -1$, find E; etc. Draw the graph, taking one small space on the horizontal line to mean 1 in. in values of e, and one small space on the vertical axis to mean 1 square *foot* in values of E, *i.e.* 144 sq. in. Find e if $E = 361$; if $E = 724$; if $E = -359$.

18. Let the error in measuring the radius of a circle whose radius is really 7 ft. long be denoted by e (in inches), and the error in the computed area by E (in square feet). Taking $\pi = 3\frac{1}{7}$ (see table at back of book), show that $E = 528\,e + 3\frac{1}{7}\,e^2$. Find E for various values of e; plot the figure; find e if $E = -524\frac{6}{7}$.

REVIEW EXERCISES XXI: CHAPTER IV

Perform the indicated multiplications; check:

1. $(a^3 - 3a^2b + 5ab^2 - b^3)(a^2 - ab + b^2)$.
2. $(16k^2 - 4k + 1)(16k^2 + 4k + 1)$.
3. $(1 - a - b)(1 + a + b + a^2 + 2ab + b^2)$.
4. $(a + 2b + 3c)(b + 2c + 3a)(c + 2a + 3b)$.
5. $(y - z)(z - x)(x - y)$.
6. $(a^4 + a^3 + a^2 + a + 1)(a - 1)$.
7. $(m^{3n} + m^{2n} + m^n + 1)(m^n - 1)$.
8. $(ax^2 + bx + c)(lx^2 + mx + n)$.
9. $-a^b b^c c^d \times a^c b^d c^a \times -a^d b^a c^b$.
10. $(10 - 3)(5 + 7)$; $(6 - 9)(8 - 10)$; $(7 - 11)(3 + 6)$.

Perform the indicated divisions (inexact divisions indicated by *, as before); check each exercise:

11. $\dfrac{z^5 - z^4 + 8z^3 - 11z^2 + 21z}{z^3 - 2z^2 + 3z}$.
12. $\dfrac{x^2 + (a + b)x + ab}{x + a}$.
13. $\dfrac{-x^6 + 3x^5 + 2x^4 - x^3 - 14x^2 - 9x + 18}{x^2 - 5x + 6}$.
14. $\dfrac{x^3 - (a + b + c)x^2 + (bc + ca + ab)x - abc}{x^2 - (a + b)x + ab}$.

* 15. $\dfrac{x^2 + px + q}{x - a}$. * 16. $\dfrac{x^3 + px^2 + qx + r}{x - a}$. 17. $\dfrac{a^6 - b^6}{a + b}$.

18. $\dfrac{y^6 + 6y^5z + 15y^4z^2 + 20y^3z^3 + 15y^2z^4 + 6yz^5 + z^6}{y^2 + 2yz + z^2}$.
19. $\dfrac{2m^7 + 3n^7 + 6m^6n - 3mn^6 + m^2n^5 + 3m^4n^3 + 3m^3n^4}{2m^3 + 4m^2n - 6mn^2 + 3n^3}$.

* 20. $\dfrac{x^4 + 3x^3 + 10x^2 + 14x + 10}{x^2 + 2x}$. * 21. $\dfrac{p^5 + q^5}{p - q}$.

Factor the following expressions:

22. $x^2+10x+24$.

23. $x^2-10x+24$.

24. $x^2+10x-24$.

25. $x^2-10x-24$.

26. $x^2+2ax+(a^2-b^2)$.

27. a^6-b^3.

28. $2x^2+2x-40$.

29. $6x^2+x-35$.

30. $63A^2+AB-12B^2$.

31. z^4-625.

32. a^4+5a^2+9.

33. r^4+10r^2+9.

34. r^9-z^6.

35. $x^{3m}-y^{3n}$.

36. $p^5-5p^4q+10p^3q^2-10p^2q^3+5pq^4-q^5$.

37. $r^4-12r^3+54r^2-108r+81$.

38. $(n^4+3n^2+4)=(n^4+4n^2+4)-n^2$.

39. $abx^{2m}-(a^2+b^2)x^my+aby^2$.

40. $x^6-14x^3+49-y^2+2yz-z^2$.

In the following exercises solve both from the graph and from the equation, unless otherwise directed:

41. A page is to have a margin of 1 inch, and to contain 35 square inches of printing. How large must the page be if the length is to exceed the width by 2 inches?

42. How wide a path may be laid out inside the margin of a park area 100×250 feet, in order that the space taken up by the path may be 3400 square feet?

43. Such a problem as Ex. 42 would arise as follows: how wide a path will take up an area *not greater than* 4000 square feet, approximately. Solve this problem *graphically*.

Note that we are not yet in a position to solve the problem even approximately by means of the equation.

44. For a kindergarten gift, a rectangular box with a square top is to be constructed 5 inches high. If exactly 2 square feet of colored paper is used to cover the box, including the top and the bottom, what must be the length of one side?

45. In the preceding exercise, it is desired that *not more* than 5 square feet of paper be used to cover the box. How large may the side be chosen? Solve only graphically.

46. "Think of a number, double it, add three, and square the result; from this result subtract seven, and halve the remainder." The result is 21; what was the number chosen?

47. Construct a puzzle similar to the above, leading to an easy equation, involving the square of the unknown number; propound it to a friend and find the number chosen by him.

48. Find two numbers whose sum is 14 and whose product is 48.

49. The hypotenuse of a certain right-angled triangle is ten feet long; a piece of string which is exactly one fourth as long as one of the perpendicular sides is found to be exactly one third as long as the other. How long is the string, and how long is each of the perpendicular sides? (See Tables.)

50. What radius must a cone whose slant height is 5 inches have in order to be covered by 14π square inches of tinfoil?

51. What is the error E (in square inches) in the computed value of the area of a square 50 ft. long, caused by an error e (in inches), in measuring the side? Plot the graph as in Ex. 17, p. 112. Find e if $E = 2404$; if $E = -3591$.

52. How carefully must the side of the square in Ex. 51 be measured in order to be sure that the error in the computed area is not more than 2404 sq. in.? How carefully must each mark be made if the side is measured with a foot rule in the ordinary way?

53. Answer similar questions for $E = 6025$ in Exs. 51 and 52. Is it reasonable to suppose that the area can be computed (from measurement with a foot rule) to within 10 sq. ft.?

54. How carefully must one measure the radius of a circle whose radius is 21 ft. in order that the computed area may be correct to within $795\frac{1}{7}$ sq. in.? Is it reasonable to suppose that the area can be computed (from measurement of the radius with a foot rule) to within 5 sq. ft.? (See Ex. 18, p. 112.)

SUMMARY OF CHAPTER IV: MULTIPLICATION AND DIVISION; FACTORING; APPLICATIONS; pp. 67–115

Polynomial: letters occur in simple powers; $a + bx + cx^2 + \cdots + lx^n$; $a + bx + cy + dx^2 + exy + fy^2 + \cdots$. § 55, pp. 88–89.

Review Exercises for Part II of Chapter IV: Exercises XI. pp. 89–90.

PART III. SPECIAL MULTIPLICATIONS; FACTORS; TYPE FORMS. pp. 89–102.

Form I: $(x + y)^2 = x^2 + 2xy + y^2$; polynomial factors of polynomials only. Exercises XII. §§ 56–57, pp. 91–93.

Form II: $(x - y)^2 = x^2 - 2xy + y^2$. Exercises XIII. § 58, pp. 93–94.

Form III: $(x - y)(x + y) = x^2 - y^2$. Exercises XIV. § 59, pp. 94–95.

Form IV: $(x + a)(x + b) = x^2 + (a + b)x + ab$. Exercises XV. § 60, pp. 95–97.

Form V: $ax^2 + bx + c$; factors by trial; cross products. Exercises XVI. § 61, pp. 98–99.

Other Forms: $a^3 \pm b^3$, $a^4 - b^4$, $(a \pm b)^3$, $(a \pm b)^4$. Exercises XVII. § 62, pp. 99–100.

Factoring by Grouping: $a(b + c) = ab + ac$; review of § 51; rearrangements. Exercises XVIII. § 63, pp. 100–101.

Review Exercises for Part III of Chapter IV: Factoring; Exercises XIX. p. 102.

PART IV. APPLICATIONS; ENGLISH TRANSLATED INTO ALGEBRA. pp. 103–115.

Expression of English in Formulas: search for *structure* in examples. § 64, pp. 103–105.

Simple Changes in Equations: any operation that gives a *single* result may be performed on the *whole* of each side. § 65, pp. 105–106.

If a product is zero, at least one factor is zero: illustrative problems stated in English. § 66, pp. 106–109.

Solution of Problems: complete check advisable. Exercises XX. § 67, pp. 109–112.

Review Exercises for Chapter IV: Exercises XXI. pp. 113–115.

CHAPTER V. FRACTIONS

PART I. COMMON FACTORS; REDUCTION OF FRACTIONS

68. Introduction. In dealing with fractions we shall use the notation and the rules of elementary arithmetic, which we shall review.

A **fraction** indicates the quotient obtained by dividing one number, called the **numerator**, by another number, called the **denominator**. The fraction is written by placing the numerator over the denominator with a horizontal line between them.

The numerator and denominator are called the **terms** of the fraction. The form in which a fraction is written may be changed in various ways without changing the value of the fraction, as in elementary arithmetic.

Thus, $\frac{20}{12}=\frac{5}{3}$, for $\frac{20}{12}=\frac{5\cdot 4}{3\cdot 4}$, and the numerator and denominator may each be divided by this common factor 4.

Similarly, $\frac{90}{84}=\frac{2\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 3\cdot 7}=\frac{15}{14}$; here both terms of the fraction are divided by the common factor 2 and also by the common factor 3.

The rule of elementary arithmetic is:

"*The value of a fraction is unchanged if both numerator and denominator are divided by the same number.*"

We shall follow the same rule.

For example:
$$\frac{2a^2b}{4ab^2}=\frac{2\cdot a\cdot a\cdot b}{2\cdot 2\cdot a\cdot b\cdot b}=\frac{a}{2b},$$

where we have divided both numerator and denominator by 2, also by a, also by b, each of which is a common factor.

Instead of the above rule we shall say

$$\frac{C \cdot N}{C \cdot D} = \frac{N}{D},$$

which evidently means the same thing.

[C is the *common factor* of numerator and denominator of the given fraction; N is the *numerator* of the resulting fraction; D is the *denominator* of the resulting fraction.]

This principle may be *proved* by means of the rules of p. 35.

For $\qquad \dfrac{N}{D} = X$, provided $N = D \cdot X$,

and $\qquad \dfrac{C \cdot N}{C \cdot D} = X$, provided $C \cdot N = (C \cdot D) \cdot X = C \cdot (D \cdot X)$.

But by IV, p. 106, the two conditional equations preceded by "provided" are equivalent, for we may divide both sides of the last by C, if *C is not zero. Hence, $\dfrac{N}{D}$ and $\dfrac{C \cdot N}{C \cdot D}$ are the same number X, provided C is not zero.* This proof need not be learned at this time.

Notice particularly that C must not be **zero.**

Thus, in substituting numbers in the place of letters in order to check an example, or for any other reason, if the numbers substituted would make one of these divisors zero, at any point in the work, they must not be used. Again, if both numerator and denominator are divided by an expression that contains *unknown letters*, we should be careful to say that the result is correct, *provided the* (unknown) *value of that expression is not zero.*

Reversing the preceding rule, we may *multiply* both numerator and denominator by the same number, except zero. Notice, however, that *adding* the same number to both numerator and denominator is *not* allowable, in general. Thus,

$$\frac{3}{4} \text{ is } not \text{ equal to } \frac{3+2}{4+2}.$$

Likewise, *subtracting* the same number from both terms is *not* allowable.

69. Common Factors. In § 68 we used any common factor of both numerator and denominator as the divisor of both. These common factors may be found by inspection in simple examples, as in § 51, p. 79.

Ex. 1. $\dfrac{20\,a^3b^4}{15\,a^4b^3}=\dfrac{4\,b\cdot 5\,a^3b^3}{3\,a\cdot 5\,a^3b^3}=\dfrac{4\,b}{3\,a}.$

Ex. 2. $\dfrac{x^2-1}{x^2+2\,x+1}=\dfrac{(x-1)(x+1)}{(x+1)(x+1)}=\dfrac{x-1}{x+1}.$

Ex. 3. $\dfrac{m^2+5\,m+4}{m^2-6\,m-40}=\dfrac{(m+1)(m+4)}{(m-10)(m+4)}=\dfrac{m+1}{m-10}.$

If all the common factors are removed, the fraction is said to be in its **lowest terms.**

The rule is the same as in elementary arithmetic:

To reduce a fraction to its lowest terms, divide both numerator and denominator by each of their common factors; in the resulting fraction the numerator and the denominator should have no common factor.

EXERCISE I: CHAPTER V

Simplify the fractions:

1. $\dfrac{18}{20}$. **3.** $\dfrac{96}{144}$. **5.** $\dfrac{2^3\cdot 3^2\cdot 5}{2\cdot 3^3\cdot 5^2}$. **7.** $\dfrac{2\,a^3b}{3\,ab^3}$. **9.** $\dfrac{-16\,a^2bx^3}{-24\,abx^4}$.

2. $\dfrac{65}{70}$. **4.** $\dfrac{189}{105}$. **6.** $\dfrac{4^3\cdot 5\cdot 6^2}{2\cdot 3\cdot 4}$. **8.** $\dfrac{XY^2Z^3}{XY^3Z}$. **10.** $\dfrac{35\,m^2n^3p}{21\,m^3np^4}$.

11. $\dfrac{16\,m^3n^2}{24\,mp^2}$. **12.** $\dfrac{-12\,z^3t}{-30\,xyzt^2}$. **13.** $\dfrac{-10\,x^2yz^3}{2\,x^3y^2z}$. **14.** $\dfrac{p^2q-pq^2}{p^2q^2}$.

15. $\dfrac{27\,(a+b)^2(x+y)^3r^5}{15\,(a+b)^3(x+y)\,r^6}$. **16.** $\dfrac{65(q-r)(r-p)(p-1)^3}{26\,(q-r)^2(r-p)^3(p-1)}$.

70. Highest Common Factor. As in elementary arithmetic, the product of all the common factors is called the **highest common factor,** usually abbreviated **H. C. F.**

This is often called **highest common divisor, H. C. D.**; also, in elementary arithmetic, greatest common divisor, *G. C. D.*

In arithmetic we found a number as *G. C. D.* Thus the G. C. D. of 60 and 72 is 12, for $60 = 2 \cdot 2 \cdot 3 \cdot 5$ and $72 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$. The common factors are 2, 2, and 3, and their product is 12.

Just here we are working with *polynomials;* that is, expressions each of whose terms contains only simple powers of the important letters. (See §§ 55, 56, pp. 88, 91.) *We shall use as common factors of two such polynomials only factors which are themselves polynomials.** The H. C. F. here is therefore a *polynomial*, though we shall later have examples of another kind.

EXERCISES II: CHAPTER V

Reduce to lowest terms after finding the H. C. F. of numerator and denominator:

1. $\dfrac{15\,x^2y - 10\,xy^2}{5\,xy^2}$.

2. $\dfrac{12\,a^3b^2c + 18\,abc^3 + 15\,bc^2}{30\,abc}$.

3. $\dfrac{8\,pq}{16\,p^3q^2 - 24\,pq^3}$.

4. $\dfrac{m^2n - mn^2}{m^2 - n^2}$.

5. $\dfrac{z^2 - 5\,z + 6}{z^2 + z - 6}$.

6. $\dfrac{z^2 - 5\,z + 6}{z^2 - z - 6}$.

7. $\dfrac{1 - 2\,r - 8\,r^2}{1 - 7\,r - 18\,r^2}$.

8. $\dfrac{x^2 - 2\,xy + y^2}{x^2 + 4\,xy - 5\,y^2}$.

9. $\dfrac{m^6 - n^2}{m^6 + 2\,m^3n - 3\,n^2}$.

10. $\dfrac{r^2 - 2\,rs - 15\,s^2}{r^2 + 8\,rs + 15\,s^2}$.

11. $\dfrac{u^2 + 9\,uv - 10\,v^2}{2\,u^2 - uv - v^2}$.

12. $\dfrac{x^2 - 2\,x - 35}{2\,x^2 - 5\,x - 63}$.

13. $\dfrac{1 - x - 6\,x^2}{3 - x - 24\,x^2}$.

14. $\dfrac{x^2 - y^2}{x^3 - y^3}$.

15. $\dfrac{6\,v^2 - v - 2}{9\,v^2 - 4}$.

16. $\dfrac{3\,z^2 - 13\,z + 12}{9\,z^2 - 24\,z + 16}$.

17. $\dfrac{5\,s^2 + st - 6\,t^2}{s^3t - st^3}$.

18. $\dfrac{12\,x^2 - xy - 6\,y^2}{8\,x^2 - 2\,xy - 3\,y^2}$.

19. $\dfrac{5\,p^2 - 26\,pq + 5\,q^2}{3\,p^2 - 5\,pq - 50\,q^2}$.

20. $\dfrac{a^2 + 2\,ab + b^2 - c^2}{a^2 + b^2 + c^2 + 2\,bc + 2\,ca + 2\,ab}$.

---

* It should be noticed that an expression of even a single term *may* be a polynomial; it is so, provided it contains only simple powers of the important letters, or if it is a known constant. (See § 9, p. 9, and § 55, p. 88.)

71. Practical Work in Common Factors. The chief use of the H. C. F. is in the reduction of fractions. In cases where the factors cannot be found by inspection, the product of all those common factors which we *can* find is used in its place. Thus, practically speaking, we find the H. C. F. only in the simpler examples.

Ex. 1. Find the H. C. F. of 36,285 and 44,895.

We may notice that 5 and 3 are each common factors (by trial). Hence 15 is a common factor. The student might try for some time without finding any other common factor. We should say, after a reasonable amount of work, that 15 is a common factor of 36,285 and 44,895, and that we cannot easily find any others. However, 41 is also a common factor; the H. C. F. is really $15 \times 41 = 615$.

There is a long process that is sometimes used to make sure of the H. C. F.; this is hardly worth while at present, however, for the work required in it is longer than the value of the results justify. The process is found at the end of this book. (See Appendix, § 19.)

Practically speaking, we wish to reduce a fraction only so far as is convenient. For example, $\frac{36285}{44895} = \frac{2419}{2993}$ after taking out the common factors 3 and 5 which are easy to find. If we wish really to divide numerator by denominator to find the value in decimals, it is worth while to find such easy common factors as 3 and 5 and remove them first. But it would be a waste of time to try past 11 or 13 in seeking common factors by trial. Certainly it would be a waste of time to try a number as high as 41 — which is really the next common factor, for the whole work required to divide 2419 by 2993 (to several places of decimals) is *less than* the work required to find the factor 41, either by trial or by the long process mentioned above (Appendix).

We shall say that a fraction is reduced *as low as is practicable*, or is in its *lowest practicable terms*, when all the common factors of numerator and denominator that can be found by methods we know at present have been removed.

Thus: $$\frac{36285}{44895} = \frac{3 \cdot 5 \cdot 2419}{3 \cdot 5 \cdot 2993} = \frac{2419}{2993} \text{ (as low as practicable),}$$

although $$\frac{2419}{2993} = \frac{41 \times 59}{41 \times 73} = \frac{59}{73} \text{ (lowest terms).}$$

Likewise, $\dfrac{a^7b + a^5b^3 + a^3b^5}{a^5b^3 - 4\,a^4b^4 + 4\,a^3b^5 - 3\,a^2b^6} = \dfrac{a^3b(a^4 + a^2b^2 + b^4)}{a^2b^3(a^3 - 4\,a^2b + 4\,ab^2 - 3\,b^3)}$

$$= \frac{a(a^4 + a^2b^2 + b^4)}{b^2(a^3 - 4\,a^2b + 4\,ab^2 - 3\,b^3)}$$

(as low as practicable),

although $\dfrac{a(a^4 + a^2b^2 + b^4)}{b^2(a^3 - 4\,a^2b + 4\,ab^2 - 3\,b^3)} = \dfrac{a(a^2 - ab + b^2)(a^2 + ab + b^2)}{b^2(a^2 - ab + b^2)(a - 3\,b)}$

$$= \frac{a(a^2 + ab + b^2)}{b^2(a - 3\,b)} \text{ (\textit{lowest} terms).}$$

Unless the factors happen to be known, the second operation for finding the result in *lowest* terms cannot be done without considerable work; it is scarcely worth while, the result in lowest practicable terms being the one most often used. *

EXERCISES III : CHAPTER V

Reduce to the lowest terms practicable:

1. $\frac{2046}{2418}$. **2.** $\frac{3315}{4845}$. **3.** $\frac{8246}{13206}$. **4.** $\frac{41710}{43430}$. **5.** $\frac{17384}{18040}$.

6. $\frac{9636}{3852}$. **7.** $\dfrac{a^4 - b^4}{a^3 - b^3}$. **8.** $\dfrac{a^3 + b^3}{a^5 + b^5}$. **9.** $\dfrac{b^4 + 7\,b^2 - 8\,b}{b^4 + 7\,b^3 - 8\,b^2}$.

10. $\dfrac{m^3n + 3\,m^2n^2 - 4\,mn^3}{2\,m^4 + 3\,m^2n^2 - 5\,mn^3}$. **13.** $\dfrac{x^3 - x^2y - xy^2 + y^3}{x^3 + x^2y - xy^2 - y^3}$.

11. $\dfrac{r^4 - 6\,r^2 + 5}{r^2 - 6\,r + 5}$. **14.** $\dfrac{x^6 - y^6}{x^3 - 2\,x^2y + 2\,xy^2 - y^3}$.

12. $\dfrac{x^3y - xy^3}{2\,x^5y - 5\,x^3y^3 + 3\,xy^5}$. **15.** $\dfrac{3\,a^5b^2 + 6\,a^3b^4 + 27\,ab^6}{2\,a^5b^2 - 2\,a^4b^3 + 2\,a^3b^4 + 6\,a^2b^5}$.

[Many of the above exercises will of course be reduced readily to the actual *lowest* terms.]

---

* The question as to when any fraction really *is* reduced as low as is practicable is to some extent arbitrary, and depends on the breadth of the student's experience in working with literal expressions and recognizing factors. If the factorization of $a^4 + a^2b^2 + b^4$ and $a^3 - 4\,a^2b + 4\,ab^2 - 3\,b^3$, used above, were familiar, then the fraction would be in lowest terms practicable only when written in the form $\dfrac{a(a^2 + ab + b^2)}{b^2(a - 3\,b)}$. The teacher will be able to judge what degree of reduction may be reasonably expected in the case of his own students. See also Appendix, § 34, *et seq.*

PART II. RULES OF OPERATION

72. Rules of Signs. A fraction is merely an indicated quotient, as in elementary arithmetic. The rules for signs hold as in division.

Thus, $\frac{-a}{b} = -\frac{a}{b}$; $\frac{a}{-b} = -\frac{a}{b}$; and $\frac{-a}{-b} = +\frac{a}{b}$.

Likewise, $-\left(\frac{-a}{b}\right) = +\frac{a}{b}$; $-\left(\frac{a}{-b}\right) = +\frac{a}{b}$; $-\left(\frac{-a}{-b}\right) = -\frac{a}{b}$.

There are three important signs to be considered: the sign of the *numerator*, that of the *denominator*, that *in front of the whole fraction. Of these three important signs any two may be changed at the same time, without changing the value of the fraction.*

The sign in front of the fraction affects the total result.

Thus, in the fraction $-\frac{3a+5b}{2a-b}$, if $a = 8$ and $b = 3$, we get

$$-\frac{3\cdot 8+5\cdot 3}{2\cdot 8-3} = -\frac{24+15}{16-3} = -\frac{39}{13} = -3.$$

To change the sign of the numerator (or denominator) be careful to *change the sign of each term, including the first term:*

$$-\frac{3a+5b}{2a-b} = +\frac{-(3a+5b)}{2a-b} = \frac{-3a-5b}{2a-b}.$$

Check: Let $a = 8$, $b = 3$ in this answer;

$$\frac{-3\cdot 8-5\cdot 3}{2\cdot 8-3} = \frac{-24-15}{16-3} = \frac{-39}{13} = -3.$$

Again, $$-\frac{3a+5b}{2a-b} = \frac{3a+5b}{-(2a-b)} = \frac{3a+5b}{-2a+b} = \frac{3a+5b}{b-2a}.$$

Check: Let $a = 8$, $b = 3$; $\frac{3\cdot 8+5\cdot 3}{3-2\cdot 8} = \frac{24+15}{3-16} = \frac{39}{-13} = -3.$

Note, in the last check, the fraction $\frac{24+15}{3-16}$; this may also be written $\frac{-24-15}{-3+16}$, where we have changed the signs of both numerator and denominator.

If the numerator (or the denominator) is the product of several factors, a change of sign of *one* of these factors changes the sign of the whole fraction. Thus,

$$-\frac{(3-x)(x-5)}{12}=\frac{-(3-x)(x-5)}{12}=\frac{(x-3)(x-5)}{12}.$$

EXERCISES IV: CHAPTER V

Make each of the three possible simultaneous double changes of sign in each of the following fractions; check each exercise by inserting numerical values:

1. $-\frac{5}{6}$.

2. $\frac{x^2-y^2}{y^2-z^2}$.

3. $\frac{-v}{p-q}$.

4. $\frac{-ab}{b^2-c^2}$.

5. $-\frac{3a-2b}{-4z+abc}$.

6. $-\frac{abc-s^2}{u+v}$.

7. $-\frac{5xy}{-y+z}$.

8. $-\frac{k^2-l^2}{rs-t^2}$.

9. $-\frac{t-u+v-w}{-t-v+w}$.

10. $\frac{-(4x-2)(3-5x)}{5-2x}$.

11. Simplify $-\frac{-1}{-(-1)}$.

12. $\frac{(a-b)(b-c)}{(c-d)(e-f)}$.

13. Simplify $-\frac{(5-3)(2-7)}{(4-14)(8-5)}$.

73. Addition and Subtraction of Fractions. To add several fractions we proceed as in elementary arithmetic.

Thus, $\frac{2}{3}+\frac{4}{5}=\frac{10}{15}+\frac{12}{15}=\frac{22}{15}$.

Likewise, $\frac{a}{b}+\frac{c}{d}=\frac{a\cdot d}{b\cdot d}+\frac{c\cdot b}{d\cdot b}=\frac{ad+bc}{bd}$.

First reduce the given fractions to a common denominator; then add the new numerators and place this sum over the common denominator.

This rule may be proved by means of the principles on p. 35, as follows. This proof need not be learned at this time.

$$\frac{a}{b} = h \text{ if } a = hb; \ \frac{c}{d} = k \text{ if } c = k \cdot d;$$

then, $ad = h \times bd$ and $cb = k \times bd$; hence, $ad + cb = (h + k)bd$ or,

$$\frac{ad + cb}{bd} = h + k = \frac{a}{b} + \frac{c}{d}.$$

Ex. 1. $2\,xy + \frac{4\,x^2}{y^2} - \frac{3\,y}{2\,x}$, that is, $\frac{2\,xy}{1} + \frac{4\,x^2}{y^2} - \frac{3\,y}{2\,x}$.

Taking the common denominator $2\,xy^2$, we may write the preceding expression in the form:

$$\frac{2\,xy \times 2\,xy^2}{1 \times 2\,xy^2} + \frac{4\,x^2 \times 2\,x}{y^2 \times 2\,x} - \frac{3\,y \times y^2}{2\,x \times y^2} = \frac{4\,x^2y^3}{2\,xy^2} + \frac{8\,x^3}{2\,xy^2} - \frac{3\,y^3}{2\,xy^2}$$

$$= \frac{4\,x^2y^3 + 8\,x^3 - 3\,y^3}{2\,xy^2}.$$

Ex. 2. $x + 1 + \frac{x-1}{x+1} - 2 \cdot \frac{x+1}{x-1}$

$$= \frac{(x+1)\cdot(x^2-1)}{1\cdot(x^2-1)} + \frac{(x-1)\cdot(x-1)}{(x+1)\cdot(x-1)} - \frac{2(x+1)\cdot(x+1)}{(x+1)\cdot(x-1)}$$

$$= \frac{(x^3 + x^2 - x - 1) + (x^2 - 2\,x + 1) - 2\,(x^2 + 2\,x + 1)}{x^2 - 1}$$

$$= \frac{x^3 + x^2 - x - 1 + x^2 - 2\,x + 1 - 2\,x^2 - 4\,x - 2}{x^2 - 1} = \frac{x^3 - 7\,x - 2}{x^2 - 1}.$$

In these examples one of the fractions has the sign $-$. This amounts to subtracting it from the others; in any case the result is called the algebraic sum. As in example 1, if a term is inserted to be added to the others which itself is not written as a fraction ($2\,xy$ in Ex. 1), such a term may be written *in the form of a fraction* $\left(\frac{2\,xy}{1}, \text{ for example}\right)$ by inserting the denominator 1.

Such a form as $2\,xy + \frac{4\,x^2}{y^2} + \frac{3\,y}{2\,x}$ is often called a *mixed* expression, since part is in fractional form and part is not.

As common denominator, any convenient expression may be taken of which each given denominator is a factor; in any case the product of all the given denominators will surely suffice. See also § 74, below.

EXERCISES V: CHAPTER V

1. $\dfrac{x-1}{3}+\dfrac{2x+3}{2}.$

2. $\dfrac{3x+1}{2}-(x+1)-\dfrac{4x+3}{4}.$

3. $\dfrac{1}{a}+\dfrac{1}{b}.$

4. $\dfrac{1}{x+a}+\dfrac{1}{x-a}.$

5. $\dfrac{y-z}{3x}+\dfrac{2z-5x}{7x}+\dfrac{x-y}{x}.$

6. $\dfrac{x+3}{x}-\dfrac{x+4}{x+1}.$

7. $\dfrac{1}{a}-\dfrac{1}{b}.$

8. $\dfrac{1}{x+a}-\dfrac{1}{x-a}.$

9. $\dfrac{x+2}{x+3}+\dfrac{x+1}{x}-\dfrac{2(x^2+3x+1)}{x^2+3x}.$

[The additions worked out in Exs. 3, 4, 7, 8 are very often useful, and are worth committing to memory.]

10. $\dfrac{x}{x+2}-\dfrac{x+1}{x-2}-\dfrac{2}{x^2-4}.$

11. $\dfrac{1}{w}-\dfrac{1}{1+w}-\dfrac{w}{w^2}.$

12. $\dfrac{y-z}{x+y}-\dfrac{y-z}{y+z}-1.$

13. $\dfrac{1}{y-z}+\dfrac{1}{z-x}+\dfrac{1}{x-y}.$

14. $\dfrac{x}{y+z}+\dfrac{x+y+z}{y-z}-\dfrac{2y(x+y+z)}{y^2-z^2}+1.$

15. $\dfrac{3}{5-x}-\dfrac{1}{2-x}+\dfrac{2x}{10-7x+x^2}.$

74. Common Multiples. In adding fractions above, we have used as common denominator any convenient expression of which each given denominator is a factor. Such an expression is often called a **common multiple** of the given denominators. In general, a **multiple** of any expression, just as in arithmetic, is an expression of which the given expression is a factor, or, in other words, an

"exact" divisor. We refer here to *polynomials*, and we seek a common multiple of which each of the given *polynomials* are factors (see §§ 55, 70, pp. 88, 121); similarly in *arithmetic*, if *integers* are given, we seek a common multiple that is an *integer*.

The product of several expressions is evidently a common multiple of them; for each of them is a factor of their product.

Often some simpler common multiple can be found.

Thus, given $72 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$, and $39 = 3 \cdot 13$, and $48 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3$, *one* common multiple is the product of $72 \times 39 \times 48$. A simpler one is found by omitting in this product all repeated factors that occur in two or more of the numbers. Thus, $(2 \cdot 2 \cdot 2 \cdot 3 \cdot 3) \times (13) \times (2)$ is a common multiple; indeed is the **least common multiple**, *L. C. M.*, of the given numbers, *i.e.* it is the least number of which each given number is a factor.

Likewise, given $a^2 - b^2$, $4(a + b)$, and $(a - b)^2$, one common multiple is their product:

$$[a^2 - b^2][4(a + b)][(a - b)^2].$$

But $a^2 - b^2 = (a - b)(a + b)$, $(a - b)^2 = (a - b)(a - b)$, and $4(a + b) = 4(a + b)$.

Hence, a common multiple is

$$[(a - b)(a + b)][4][a - b],$$

where, just as before, we have omitted duplicate factors. This is the **lowest common multiple** * (**L.C.M.**), i.e. *if any of the factors now remaining in the result were omitted, the result would not be a common multiple.*

The rule is precisely like that of elementary arithmetic: *Factor each of the given expressions; write as their* **L.C.M.** *the product formed by omitting in each any of its factors*

* It is usual in arithmetic to use the word "least," in algebra the word "lowest," in this connection.

that is already written down as many times as it occurs in the expression under consideration; in short, **omit the duplicate factors.** Care must be taken to have a factor occur *in the* **L. C. M.** *as many times as it appears in any one factor.*

Practically it may not be convenient to factor the given expressions completely; in this case we omit only those duplicate factors which can be found by the methods known at present. The lowest practicable multiple is thus found; it is the simplest common multiple that can be found conveniently.

Thus, given 44,895 and 36,285, we find,

$$44{,}895 = 3 \cdot 5 \cdot 2993.$$
$$36{,}295 = 3 \cdot 5 \cdot 2419.$$

Hence, the lowest practical multiple is

$$(3 \cdot 5 \cdot 2993) \times (2419) = 108{,}601{,}005.$$

As a matter of fact (see p. 122) 41 is a factor of 2993 and also of 2419. Hence the **L. C. M.** is

$$(3 \cdot 5 \cdot 41 \cdot 73) \times (59) = 2{,}648{,}803.$$

In this example it is clear that

$$\text{L. C. M.} = \frac{\text{Product of the } two \text{ given expressions}}{\text{H. C. F.}}.$$

This is true if *only two* numbers are given, for the H. C. F. is exactly the product of those duplicate factors that are to be omitted in forming the L. C. M.

This remark leads (see Appendix, § 19) to a general method for finding L. C. M. in any case; but this process is usually long and is scarcely justified by the value of the result.

In adding fractions by the rule of § 73, *p.* 125, *choose as the common denominator the lowest practical multiple of the given denominators; then reduce each fraction to this denominator by multiplying its numerator and its denominator by the quotient formed by this common denominator divided by the given denominator, and proceed as on p.* 125.

Ex. 1. $\dfrac{2xy}{x^2-y^2}-\dfrac{2x-y}{x-y}-\dfrac{x^2+2y^2}{(x+y)^2}$.

We notice that the L. C. M. of the denominators is

$$(x-y)(x+y)(x+y)=(x-y)(x+y)^2.$$

Hence, the expression given

$$=\frac{2xy(x+y)}{(x-y)(x+y)^2}-\frac{(2x-y)(x+y)^2}{(x-y)(x+y)^2}-\frac{(x^2+2y^2)(x-y)}{(x-y)(x+y)^2}$$

$$=\frac{(2x^2y+2xy^2)-(2x^3+3x^2y-y^3)-(x^3-x^2y+2xy^2-2y^3)}{(x-y)(x+y)^2}$$

$$=\frac{-3x^3+3y^3}{(x-y)(x+y)^2}=-\frac{3(x^3-y^3)}{(x-y)(x+y)^2}$$

$$=-\frac{3(x-y)(x^2+xy+y^2)}{(x+y)(x+y)(x-y)}=-\frac{3(x^2+xy+y^2)}{(x+y)^2}.$$

In this example the result is reduced to its lowest terms; the student should *try* to do this in every example, and *the result should be reduced to lowest terms or at least as far as is practicable.* The student is advised to leave the common denominator in *factored form*, as in the preceding example, until the work is completed.

EXERCISES VI: CHAPTER V

Perform the indicated operations:

1. $\dfrac{1}{a+a^2}-\dfrac{1}{a-a^2}$.

2. $\dfrac{x}{x^2+xy}-\dfrac{x-y}{x^2+2xy+y^2}$.

3. $\dfrac{1}{x^2+2x-15}+\dfrac{1}{x^2-x-6}$.

4. $\dfrac{x-1}{x^2+2x-15}-\dfrac{x-4}{x^2-x-6}$.

5. $\dfrac{x+y}{x^2+xy+y^2}+\dfrac{1}{x-y}-\dfrac{xy+2y^2}{x^3-y^3}$.

6. $\dfrac{y-z}{y(y+z)}-\dfrac{y+z}{y(y-z)}+\dfrac{4z}{y^2-z^2}$.

7. $\dfrac{1}{(x-y)(x-z)}+\dfrac{1}{(y-z)(y-x)}+\dfrac{1}{(z-x)(z-y)}.$

8. $\dfrac{2r-1}{r^2+2r-3}-\dfrac{r-2}{r^2-1}-\dfrac{1}{r+3}.$

9. $\dfrac{x-1}{x^2-5x+6}-\dfrac{x-1}{x^2-x-2}-\dfrac{4}{x^2-2x-3}.$

10. $\dfrac{1}{x^2-3x-4}+\dfrac{1}{x^2-5x+4}.$

11. $\dfrac{2x}{x^2+y^2}+\dfrac{1}{x+y}-\dfrac{1}{y-x}-\dfrac{4x^3}{x^4-y^4}.$

12. $\dfrac{2x}{2x+3}-\dfrac{1-x}{2-x}+\dfrac{7}{2x^2-x-6}.$

13. $\dfrac{2}{x^2-2x+1}-\dfrac{3}{x^3-3x^2+3x-1}.$

14. $\dfrac{5}{3x+2}+\dfrac{1}{1-2x}-\dfrac{7(x-1)}{6x^2+x-2}.$

15. $\dfrac{x+1}{2x^2+5x+2}-\dfrac{2x+1}{4x^2+5x-6}+\dfrac{3}{8x^2-2x-3}.$

75. Multiplication. We have already found and stated the rules for multiplying fractions (see p. 67); these are

$$\frac{a}{b}\times c=\frac{ac}{b};\quad \frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}.$$

In words these rules read as in elementary arithmetic:

The product of two or more fractions is the product of their numerators divided by the product of their denominators.

Notice that this rule may be applied when one of the given factors is not in fractional form if we supply such a factor with a denominator 1.

A factor that appears in the numerator of any of the given factors may be canceled with the same factor in the denominator of any other one, for this amounts to dividing numerator and denominator of the result by the same factor.

Thus,

Ex. 1. $\dfrac{12}{35}\times\dfrac{25}{14}=\dfrac{\cancel{2}\cdot 6}{\cancel{5}\cdot 7}\times\dfrac{\cancel{5}\cdot 5}{\cancel{2}\cdot 7}=\dfrac{6\cdot 5}{7\cdot 7}=\dfrac{30}{49}.$

Ex. 2. $\dfrac{4\,a^2}{33\,b^3}\times\dfrac{9\,b^4}{20\,a^3}=\dfrac{\cancel{4}\cdot\cancel{a^2}}{\cancel{3}\cdot 11\cdot\cancel{b^3}}\times\dfrac{\cancel{3}\cdot 3\cdot b\cdot\cancel{b^3}}{\cancel{4}\cdot 5\cdot a\cdot\cancel{a^2}}=\dfrac{3\,b}{55\,a}.$

Ex. 3. $$-\left(\frac{x^2-1}{3\,x+2}\right)\times\left(\frac{x-1}{x+1}\right)=-\frac{(x-1)(x+1)}{3\,x+2}\cdot\frac{x-1}{x+1}$$

$$=-\frac{(x-1)^2}{3\,x+2}.$$

Do not fail to take account of the rule of signs in multiplying.

The result should be reduced to *lowest terms* or as nearly to lowest terms as is practicable.

EXERCISES VII : CHAPTER V

Perform the following multiplications:

1. $\frac{16}{21}\times\frac{49}{32}.$ **2.** $\frac{30}{77}\times\frac{22}{45}.$ **3.** $-\dfrac{169}{-72}\times\dfrac{-90}{91}.$

4. $\dfrac{18}{-35}\times-\dfrac{15}{-22}\times\dfrac{-77}{27}.$ **8.** $\frac{2}{3}\times\frac{4}{5}\times\frac{6}{7}\times\frac{8}{9}.$

5. $\dfrac{15\,x^2y}{8\,z^2}\times\dfrac{16\,yz}{45\,x}.$ **9.** $\dfrac{35\,b^3c^3}{-12\,a^2}\times\dfrac{24\,a^2b^2}{-25\,c^2}.$

6. $\dfrac{u-v}{u^2+2\,uv+v^2}\times\dfrac{v+u}{v-u}.$ **10.** $\dfrac{a}{b}\times\dfrac{b}{c}\times\dfrac{c}{d}\times\dfrac{d}{a}.$

7. $\dfrac{1-t-20\,t^2}{1-t^3}+\dfrac{1-t}{1-16\,t^2}.$ **11.** $\dfrac{p-q}{p+q}\times(p^2-q^2).$

12. $\dfrac{x^2-5x+6}{x^2-5x+4}\times\dfrac{x^2-7x+12}{x^2-7x+10}$.

13. $\dfrac{a-b}{a^2-a-2}\times\dfrac{a^2+2a+1}{a^2-2ab+b^2}$.

14. $\dfrac{-15(x-a)^2(y-b)^3}{14(z-c)^2}\times\dfrac{77(x-a)(z-c)}{-25(y-b)^4}$.

15. $\dfrac{ax+x^2}{ax-x^2}\times\dfrac{x^2-a^2}{x^2+3ax+2a^2}$.

16. $\dfrac{x^3-y^3}{x^2+7xy+6y^2}\times\dfrac{x^2+4xy+3y^2}{x^2+xy+y^2}$.

17. $\dfrac{xy+y^2}{p-q}\times\dfrac{px-py-qx+qy}{x^3+y^3}$.

18. $\dfrac{a^2+b^2+c^2+2bc-2ca-2ab}{a+b-c}\times\dfrac{a^2-b^2+2bc-c^2}{a^2-b^2-2bc-c^2}$.

19. $\dfrac{x^3+x^2y+xy^2+y^3}{x^3-3x^2y+3xy^2-y^3}\times\dfrac{x^3-y^3}{x^3+y^3}$.

20. $\dfrac{4x^2+x-3}{x^2+x-2}\times\dfrac{3x^2+5x-2}{8x^2-2x-3}$.

21. $\dfrac{6a^2-a-2}{a^2-a-2}\times\dfrac{a^2-1}{4a^2-1}$.

22. $\dfrac{x^2y+xy^2}{6x^2+xy-y^2}\times\dfrac{2x^2-xy-y^2}{x^2y^2}$.

23. $\dfrac{x^4-y^4}{12x^4-x^2y^2-6y^4}\times\dfrac{3x^2+2y^2}{x^3-x^2y-xy^2+y^3}$.

24. $\dfrac{a^3+2a^2b+2ab^2+b^3}{a^3-3a^2b+3ab^2-3b^3}\times\dfrac{a^3+3a^2b+3ab^2+b^3}{a^3-b^3}$.

76. Division : Complex Fractions. We have also found and stated (see p. 74) the rule for dividing one fraction by another ; it is

$$\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}.$$

The proof consists in the fact that the quotient $\times$ the divisor = the dividend :

$$\frac{ad}{bc} \times \frac{c}{d} = \frac{a}{b}.$$

In words this rule reads as in elementary arithmetic:

To divide one fraction by another, multiply the dividend by the divisor inverted.

For $$\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc} = \frac{a}{b} \times \frac{d}{c}.$$

Ex. 1. $\dfrac{x^2-1}{3x+2} \div \dfrac{x+1}{x-1} = \dfrac{x^2-1}{3x+2} \times \dfrac{x-1}{x+1} = \dfrac{(x-1)^2}{3x+2}.$

Ex. 2.

$$\frac{x^2-1}{3x+2} \div x+1 = \frac{x^2-1}{3x+2} \div \frac{x+1}{1} = \frac{x^2-1}{3x+2} \times \frac{1}{x+1} = \frac{x-1}{3x+2}$$

In this example the divisor is a fraction only after we insert the denominator 1. Though we should not usually insert this denominator, it is often convenient to do so.

The result of inverting a fraction is called its **reciprocal.** Thus, the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$. In general, the reciprocal of any number is 1 divided by that number. The reciprocal of

$$4 \text{ is } \frac{1}{4}; \text{ of } \frac{1}{4} \text{ is } 1 \div \frac{1}{4} = 4; \text{ of } \frac{a}{b} \text{ is } 1 \div \frac{a}{b} = 1 \times \frac{b}{a} = \frac{b}{a}.$$

The division of one fraction by another is often indicated by means of the fraction form. In this case the whole is called a **complex fraction.**

Ex. 1. $$\frac{\dfrac{x^2+1}{x^2-1}}{\dfrac{x+1}{x-1}}=\frac{x^2+1}{x^2-1}\div\frac{x+1}{x-1}=\frac{x^2+1}{x^2-1}\times\frac{x-1}{x+1}=\frac{x^2+1}{(x+1)^2}.$$

Ex. 2. $$\frac{1-\dfrac{a}{a+b}}{1+\dfrac{b}{a-b}}=\frac{\dfrac{a+b}{a+b}-\dfrac{a}{a+b}}{\dfrac{a-b}{a-b}+\dfrac{b}{a-b}}=\frac{\dfrac{b}{a+b}}{\dfrac{a}{a-b}}$$

$$=\frac{b}{a+b}\div\frac{a}{a-b}=\frac{b}{a+b}\times\frac{a-b}{a}=\frac{b(a-b)}{a(a+b)}.$$

If there are more than two horizontals, great care should be taken to mark the main horizontal line heavily, for neglect to do so may lead to serious error. Thus,

$$\frac{2}{\dfrac{3}{5}}=2\times\frac{5}{3}=\frac{10}{3},\text{ while }\frac{\dfrac{2}{3}}{5}=\frac{2}{3}\times\frac{1}{5}=\frac{2}{15}.$$

Similarly, the position of the *main* horizontal line is essential in fractions in algebra, as in arithmetic; a mistake in recognizing which is the main horizontal line causes a mistake in the result.

EXERCISES VIII: CHAPTER V

What is the reciprocal of:

1. 2? $\frac{1}{2}$? -2? $-\frac{1}{2}$?

2. a? $\frac{1}{a}$? $-a$? $-\frac{1}{a}$?

3. $\frac{a}{b}$? $\frac{-a}{b}$? $\frac{a}{-b}$? $-\frac{a}{b}$?

Perform the following divisions:

4. $\frac{15}{16}\div-\frac{5}{4}$.

5. $-\frac{20}{21}\div-\frac{15}{14}$.

6. $\frac{-64}{-65}\div-\frac{40}{-39}$.

7. $\frac{144}{25}\div\frac{126}{125}$.

8. $\frac{75}{76}\div 100$.

9. $-\frac{4}{25}\div\frac{-1}{100}$.

10. $\dfrac{-12\,a^3b^2c}{35\,x^2y} \div \dfrac{24\,a^2b}{-7\,xy}$.

11. $\dfrac{36\,bc^2}{25\,a} \div \dfrac{32\,ab}{15}$.

12. $\dfrac{50\,x^3y^4}{49\,z^2} \div \dfrac{75\,x^2y}{56\,z^3}$.

13. $\dfrac{4\,m^2}{-9\,ln^3} \div \dfrac{-12\,lm}{n^2}$.

14. $\dfrac{-65\,r^2}{64\,s^2} \div \dfrac{143\,rs^3}{-144}$.

15. $\left(1 + \dfrac{4\,xy}{(x-y)^2}\right) \div \left(1 + \dfrac{xy+y^2}{x^2-y^2}\right)$.

16. $\dfrac{xy}{x^2+2\,xy+y^2} \div \dfrac{x^2+2\,xy}{x^2-y^2}$.

17. $\dfrac{a^2-2\,ab-3\,b^2}{a^2-2\,ab+b^2} \div \dfrac{a^2-b^2}{a^2b-ab^2}$.

18. $\dfrac{p^2+pq-6\,q^2}{p^2-q^2} \div \dfrac{p^2-3\,pq+2\,q^2}{p^2+pq}$.

19. $\dfrac{z^2+8\,z+15}{z^2-6\,z+8} \div \dfrac{z^2+3\,z-10}{z^2-5\,z+4}$.

20. $\dfrac{z^4-16}{z^3-1} \div \dfrac{2-z-z^2}{1+z+z^2}$.

Simplify the complex fractions:

21. $\dfrac{\frac{1}{2}+\frac{2}{3}}{\frac{2}{3}+\frac{2}{9}+\frac{1}{12}}$.

22. $\dfrac{\dfrac{x^2+x-6}{x^2-2\,x-3}}{\dfrac{x^2+2\,x-3}{x^2-3\,x}}$.

23. $\dfrac{\dfrac{6\,x^2+x-12}{2\,x^2+x-6}}{\dfrac{9\,x^2-16}{x^2+4\,x+4}}$.

24. $\dfrac{\dfrac{1}{x^2-5\,x+6}}{\dfrac{1}{2\,x^2-9\,x+10}}$.

25. $\dfrac{\dfrac{y^2-25}{y^2+7\,y+12}}{\dfrac{y^2+10\,y+25}{y^2-y-12}}$.

26. $\dfrac{\dfrac{x-1}{x+1}+\dfrac{x+1}{x-1}}{\dfrac{x+1}{x-1}\times(x^2+1)}$.

PART III. PROPORTION

77. Definitions and Introduction. If two fractions are equal, their four terms are said to be **in proportion.**

Ex 1. $\frac{3}{6}=\frac{4}{8}$, for each is equal to $\frac{1}{2}$.

This is sometimes written $3:6=4:8$, and read "3 is to 6 as 4 is to 8," but there is no advantage in doing so.

Ex. 2. $\dfrac{x^2-1}{(x-1)^2}=\dfrac{x+1}{x-1}$, for $\dfrac{x+1}{x-1}=\dfrac{x+1}{x-1}\times\dfrac{x-1}{x-1}=\dfrac{x^2-1}{(x-1)^2}$.

Likewise all equalities of two fractions given above are proportions. Any equation is a proportion if the denominator 1 is supplied in case no denominator is given.

A fraction is often called a **ratio**, and $\dfrac{a}{b}$ is written in the form $a:b$, and read "*the ratio of a to b.*"

78. Standard Changes. We may make a new proportion (equation) from an old one by the operations of § 65, p. 106. Some of the forms derived have been given names.

I. If $\dfrac{a}{b}=\dfrac{c}{d}$, we may add 1 *to each side;* then,

$$\frac{a}{b}+1=\frac{c}{d}+1, \text{ or } \frac{a+b}{b}=\frac{c+d}{d}.$$

This is called the process of **composition**.

Ex. 1. $\dfrac{2}{3}=\dfrac{4}{6}$, hence, $\dfrac{2+3}{3}=\dfrac{4+6}{6}$, or, $\dfrac{5}{3}=\dfrac{10}{6}$.

Ex. 2. $\dfrac{x-y}{x+y}=\dfrac{x^2-y^2}{x^2+2xy+y^2}$; hence, $\dfrac{2x}{x+y}=\dfrac{2x^2+2xy}{x^2+2xy+y^2}$.

II. If $\dfrac{a}{b}=\dfrac{c}{d}$, *we may subtract* 1 *from each side;* then,

$$\frac{a}{b}-1=\frac{c}{d}-1, \text{ or } \frac{a-b}{b}=\frac{c-d}{d}.$$

This is often called the process of **division**, but this name is not well chosen, since "division" means something else.

III. If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a+b}{b}=\frac{c+d}{d}$ and $\frac{a-b}{b}=\frac{c-d}{d}$.

Dividing the right-hand sides, and also the left-hand sides, we find,

$$\frac{\frac{a+b}{b}}{\frac{a-b}{b}}=\frac{\frac{c+d}{d}}{\frac{c-d}{d}}, \quad \text{or,} \quad \frac{a+b}{a-b}=\frac{c+d}{c-d}.$$

This is called the process of **composition and division**.

IV. If $\frac{a}{b}=\frac{c}{d}$, *we may divide* 1 *by each side*, if neither is zero; then

$$\frac{1}{\frac{a}{b}}=\frac{1}{\frac{c}{d}}, \quad \text{or,} \quad \frac{b}{a}=\frac{d}{c}.$$

This is called the process of **inversion**.

V. If $\frac{a}{b}=\frac{c}{d}$, *we may multiply each side by bd;* then

$$\frac{a}{b}\times bd=\frac{c}{d}\times bd, \text{ or, } ad=bc.$$

This is called the process of **clearing of fractions** or the process of **cross-multiplication**.

VI. Finally, if $ad=bc$, we may divide each side by bd, if bd is not zero, [or by dc; or by ab; or by ac], then

$$\frac{ad}{bd}=\frac{bc}{bd}; \quad \text{or,} \quad (1)\ \frac{a}{b}=\frac{c}{d};$$

$$\left[\text{or, } (2)\ \frac{a}{c}=\frac{b}{d};\ \text{ or, } (3)\ \frac{d}{b}=\frac{c}{a};\ \text{ or, } (4)\ \frac{d}{c}=\frac{b}{a}\right].$$

A comparison of these equivalent forms shows several permissible changes that are sometimes given names; thus, the change from form (1) to form (2) is called **alternation**; the change from form (1) to form (4) is called **inversion**; but the simplest way to remember all of these is to remember the operation of **clearing of fractions**, No. V above, and the principle stated in VI, which amounts to dividing both sides by the same number.

Besides these six, many other changes are allowable; for example, we might add 2 to each side, or subtract 3 from each side, etc. In general we merely carry out operations that are allowable with any equation. These principles are often valuable in solving equations.

EXERCISES IX : CHAPTER V

In the following exercises, apply to each given proportion the process referred to by Roman numerals:

1. $\dfrac{x+2}{x-2}=\dfrac{12}{16}$; I, II, III. **2.** $\dfrac{a-3}{3}=\dfrac{3}{1}$; I. **3.** $\dfrac{1}{x}=\dfrac{-5}{8}$; IV.

4. $\dfrac{x^2-2xy+y^2}{x^2+2xy+y^2}=\dfrac{9}{25}$; I, II, III, IV. **5.** $\dfrac{3}{5}=\dfrac{15}{x}$; V, VI (3).

6. $\dfrac{a+b}{a-b}=\dfrac{x-y}{x+y}$; I, II, III, IV. **7.** $\dfrac{x^2-4}{x^2+4}=\dfrac{5}{13}$; I, II, III.

8. $\dfrac{p-2q}{2p+q}=\dfrac{1}{4}$; I, II, III. **9.** $\dfrac{a-5b}{a+5b}=\dfrac{2}{1}$; I, II, III, IV, V.

10. $\dfrac{2x-1}{2x-3}=\dfrac{2}{3}$; II, V, VI (2). **11.** $\dfrac{m^2-n^2}{m^2+n^2}=\dfrac{m^3-n^3}{m^3+x^3}$; I, II, VI (2).

79. Variables in Proportion. *Proportions* (equalities of fractions) arise whenever two *varying* quantities are so related that their quotient is always the same. (See § 20, p. 25.)

Thus, if butter is 30 cents per pound, n pounds cost $30\,n$ cents, or

$$p = 30\,n,$$

where p is the cost in cents and n is the number of pounds. (See p. 23.)

If $n = 3, p = 90$; if $n = 4, p = 120$; etc.; and we have

$$\frac{90}{3} = \frac{120}{4} = \text{etc.} = \frac{p}{n} \text{ always } = 30 \text{ always.}$$

In case two varying quantities y and x have a constant quotient, *i.e.* $\frac{y}{x} = k$, where k is constant, any two pairs of values of y and x form a proportion, and we say that the variable quantity y is **proportional** to the variable quantity x; that is, any pair of values of y and x form a proportion with any other pair. We have already used such proportional quantities and have drawn corresponding figures, which we found to be straight lines. (See pp. 20–25.) We shall now make clear that *the figure is always a straight line* if $\frac{y}{x} = k$, *i.e.* if $y = k \cdot x$.

80. Graph. In dealing with an equation of the form

$$y = kx,$$

we found always a straight-line figure. (See p. 25.) We see that if

$$x = 1, \quad y = k;$$
$$x = 2, \quad y = 2\,k;$$
$$x = 3, \quad y = 3\,k;$$
$$\text{etc.},$$

and we wish to show that these points lie on a straight line. To plot the point $x = 1$, $y = k$ we go one unit to the right and k units upwards to the point marked A.

To plot the point $x = 2$, $y = 2\,k$ we go one unit *beyond* A to the right and k units above A (Fig. 20).

The rectangle whose corners are O and A is precisely the same shape and size as the rectangle whose corners are A and B. Hence, the diagonal OA has the same

direction as the diagonal AB; consequently AB is an extension of the straight line OA.

The argument is the same from B on to the next point C; from C on to D, etc. Likewise backward from O to L, thence to M, etc.

If smaller steps are taken, the argument is always the same. Thus, steps half as wide and half as high would again bring us to the same line.

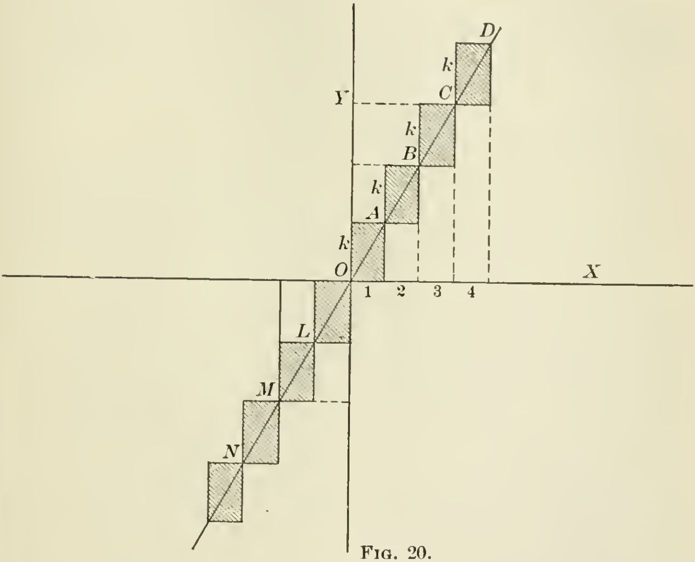


Fig. 20.

The idea of steps — of actual stair steps — is a good illustration of this; the broken line one unit to the right and k up makes such a stairway. The edges of these steps make a straight line as on any ordinary stairway.

Finally we may say, *The equation*

$$y = kx$$

is represented by a straight line through the starting point, which rises k units in passing 1 *unit to the right.*

Ex. 1. In the problem just mentioned

$$p = 30\,n,$$

where p is the price in cents of n pounds of butter. We may now draw the corresponding figure much more easily than on p. 24 where the same example was given.

If we draw *only two points* of the figure and connect these by a straight line, we know the figure is correct, since we know in advance that it is a straight line. If $n = 0$, $p = 0$ (that is, *no* pounds costs *no* money); if $n = 1$, $p = 30$; if $n = 5$, $p = 150$.

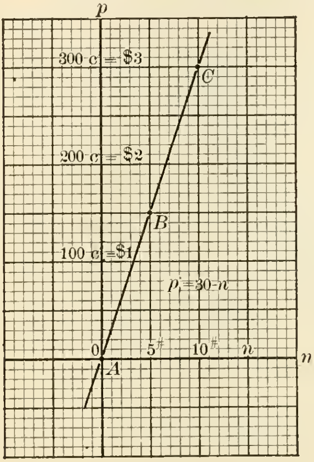


Fig. 21.

Plot the point $n = 0$, $p = 0$ at A.

Plot the point $n = 5$, $p = 150$ at B.

Join A and B by a straight line.

This straight line is the desired figure; from it can be read off at once the answers (approximately) for a variety of problems: the cost of a given number of pounds; the number of pounds that can be bought for a given amount of money, etc., as on p. 20.

Thus, having worked the two simplest examples of which we can think (that is, if $n = 0$, $p = 0$ and if $n = 1$, $p = 30$), we have in this figure the solutions (approximately) for any problems in proportion which might be given under this example.

It is well to plot a third point. Thus, if $n = 10$, $p = 300$. Plot this point at C. It lies on the straight line. *If it did not, we should know that we had made some error.* The value in plotting three points lies in this check on the accuracy of the work.

Ex. 2. In the Fahrenheit and Centigrade thermometers the scales of temperature are made in divisions that are *proportional.* Thus 9 divisions on the Fahrenheit scale are equivalent to 5 divisions on the Centigrade scale. If the temperature rises 9 degrees by a Fahrenheit thermometer, it rises 5 degrees

by a Centigrade thermometer. The amount of rise or fall on either scale can be found by ordinary proportion, in case the rise or fall is known on the other.

But the Fahrenheit scale is marked 32° at the freezing point, whereas the Centigrade is marked 0° at the freezing point. We must therefore subtract 32 from each Fahrenheit temperature before we can conveniently compare it with Centigrade temperature.

Hence, $F-32$ is the quantity to be compared with C, if F and C stand for the readings of the two thermometers at the same time and place. Since these compare as 9 compares to 5, we have

$$\frac{F-32}{C}=\frac{9}{5},$$

and we say that $F-32$ is *proportional* to C, the constant quotient (or ratio) being $\frac{9}{5}$.

Multiplying both sides by 5, also by C, we have

$$5\,(F-32)=9\,C,$$

or,

$$5\,F-160=9\,C.$$

To draw the corresponding picture, we may, as on p. 23, take this equation (which is given in the Tables without argument) and plot several points. We need only plot two of them, though three are better.

If $C=0$, $F=32$, as above (this is the freezing point).

If $C=20$, $F=68$ (this is the ordinary temperature).

If $C=100$, $F=212$ (this is the boiling point of water). Plot these points; they are L, M, N, respectively, in the figure; the straight line through these is the graph.

Notice that it is here *not* F but rather $F-32$ which is proportional to C. Hence, the *dotted* line which is just 32 points below

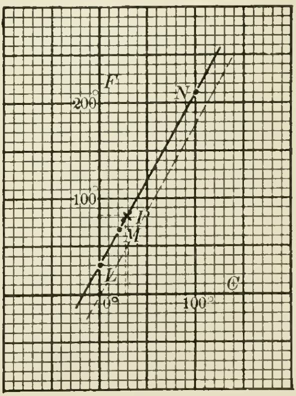


FIG. 22.

the line LMN is the line which represents the proportion of $F - 32$ and C. The heavy line really represents the corresponding values of F and C.

Given values of F, we can find the corresponding C either from the figure (approximately) or from the equation

$$5F - 160 = 9C,$$

and conversely F can be found if C is given.

Thus, if a Centigrade thermometer reads 28°, find the temperature in the ordinary (Fahrenheit) scale. In the figure we move out along the line marked C to 28, then directly upwards till we meet the straight line LMN. The height is the value of F; it is seen to be 82, approximately. From the equation,

$$5F - 160 = 9C = 9 \times 28 = 252.$$

Add 160 to each side: $5F = 252 + 160 = 412.$

Divide each side by 5: $F = 82.4.$

The *exact* answer is therefore 82°.4; notice that the answer, 82°, obtained from the figure, is not exactly correct.

In any case it is well to get a result from the figure and a result from the equation; the result from the figure is only approximately correct; that from the equation is exact. The result from the figure serves as a check on the exact answer from the equation.

In many cases an exact answer is not necessary. In the temperature example above, it is often desirable to know *approximately* the temperature; in fact, very few thermometers, except those especially made for experiments in physics, etc., are really accurate. The error made above ($\frac{4}{10}$ of a degree) is smaller than the error made by most thermometers in ordinary use. The student should in no case express an answer more exactly than the circumstances warrant; thus, no ordinary thermometer, even of those used in physical laboratories, will read correctly nearer than tenths of a degree. Hence, in such an expression as 35°.74, the last figure does not really mean anything and should be omitted.*

---

* It is suggested in the text that to plot the graph of $y = kx$, the two determining points may be chosen as (0, 0) and (1, k). This, as well as the ordinary practice of plotting $ax + by + c = 0$, by locating the points $(0, -c/b)$, $(-c/a, 0)$ is unwise as a *general* rule. The disadvantages are two in the latter case, and at least one applies to the former also:

1. The numbers that fix the points thus located may be fractions with large denominators which cannot be located accurately; proper

EXERCISES X: CHAPTER V

By the principles of § 78, solve the following equations (Exs. 1–6):

1. $\dfrac{p}{p-8}=\dfrac{9}{5}\cdot$ [Use II or V or VI (2).]

2. $\dfrac{x+6}{x-6}=\dfrac{11}{7}\cdot$ [Use II or V or VI (2).]

3. $\dfrac{x^2-3}{x^2}=\dfrac{2}{3}\cdot$

4. $\dfrac{x^2}{x+2}=\dfrac{8}{3}\cdot$

5. $\dfrac{4}{x}=\dfrac{x}{324}\cdot$

6. $\dfrac{x-2}{16}=\dfrac{2}{x+2}\cdot$

Solve the following equations (Exs. 7–10) for the letters indicated:

7. $\dfrac{x-a}{x+a}=\dfrac{1}{3}\cdot$ [Solve for x; also solve for a.]

8. $\dfrac{v+w}{w}=\dfrac{9}{4}\cdot$ [Solve for w; also solve for v.]

9. $\dfrac{x+a}{x-a}=\dfrac{c-d}{c+d}\cdot$ [Solve for x; also solve for a.]

10. $\dfrac{a-b-c}{a+b+c}=\dfrac{c}{2b+3c}\cdot$ [Solve for a.]

If $\dfrac{a}{b}=\dfrac{c}{d}$, show that the statements of Ex. 11–29 are correct; in doing so, it is justifiable to clear of fractions if no easier process is evident:

11. $\dfrac{a^2}{b^2}=\dfrac{c^2}{d^2}\cdot$

12. $\dfrac{a^2-b^2}{a^2}=\dfrac{c^2-d^2}{c^2}\cdot$

13. $\dfrac{ab}{cd}=\dfrac{b^2}{d^2}\cdot$

---

integral points might be as easily found, and much more easily plotted. *E.g.*, to plot accurately the graph of $7x+3y=10$ by locating $(1\frac{3}{7}, 0)$ and $(0, 3\frac{1}{3})$ is a practical impossibility; to do so by locating $(1, 1)$ and $(4, -6)$ is exceedingly easy.

2. The points located as above are often too close together for accurate plotting. A slight deviation in either point may throw the general direction of the line entirely off. The graph of $y=\frac{1}{10}x$ cannot be accurately plotted with equal small scales for x and y, by locating $(0, 0)$ and $(1, \frac{1}{10})$. The work can be more accurately done by locating $(0, 0)$ and $(10, 1)$; or better, $(10, 1)$ and $(-10, -1)$.

14. $\dfrac{a^2+b^2}{ab}=\dfrac{c^2+d^2}{cd}$.

19. $\dfrac{a^2-c^2}{a^2+c^2}=\dfrac{b^2-d^2}{b^2+d^2}$.

15. $\dfrac{a+nb}{b}=\dfrac{c+nd}{d}$.

20. $\dfrac{a+nb}{a}=\dfrac{c+nd}{c}$.

16. $\dfrac{pa+qb}{a}=\dfrac{pc+qd}{c}$.

21. $\dfrac{pa+qb}{ra+sb}=\dfrac{pc+qd}{rc+sd}$.

17. $\dfrac{a^2+b^2}{ac+bd}=\dfrac{ac+bd}{c^2+d^2}$.

22. $\dfrac{ax+by}{cx+dx}=\dfrac{a}{c}=\dfrac{b}{d}$.

18. $\dfrac{a^2-2ac+c^2}{b^2-2bd+d^2}=\dfrac{ac}{bd}$.

23. $\dfrac{ab-c^2}{b^2-cd}=\dfrac{a^2-cd}{ab-d^2}$.

If a, b, c, d are all different from zero, show that:

24. If $\dfrac{pa+qb}{a}=\dfrac{pc+qd}{c}$, then $\dfrac{a}{b}=\dfrac{c}{d}$.

25. If $\dfrac{a^2+b^2}{ac+bd}=\dfrac{ac+bd}{c^2+d^2}$, then $\dfrac{a}{b}=\dfrac{c}{d}$.

26. Express the simple interest, i, at 6% on \$500 for t years. Draw the graph. In how many years will the interest be \$90? \$60? What quantities are here proportional?

27. Express the *amount* in the above problem. Draw the graph on the same diagram as that for example 26. What quantities are proportional? State and explain the relative position of the two figures.

28. Compare by an equation the readings of a Centigrade and a Réaumur thermometer as given in the table at the back of the book. What quantities are proportional? Draw the graph.

Compare the readings of a Fahrenheit and a Réaumur thermometer. What quantities are proportional? Draw the graph.

29. John can run 20 feet a second; Henry, 15. Compare the distances, x, y, from the same starting point of John and Henry at any time. What quantities are proportional? Draw the graph.

30. If John gives Henry a start of 5 feet, compare the distances from John's starting point, in Ex. 29. What quantities are proportional? Draw the graph. Explain the relation to the figure for Ex. 29.

31. If John and Henry, in Ex. 30, meet, show that $y = x$ for the point of meeting. Draw the graph giving *all* values for which this relation holds. Where will John meet Henry in Ex. 30? Solve this problem also by means of the equations.

32. *Try* to find graphically, as in Ex. 31, the distance at which John and Henry meet in Ex. 29. Explain your failure.

33. State the general formula for the interest, i, in terms of the principal, p, the rate per cent, r, and the time, t, for simple interest. If the principal and time are constant, to what is the interest proportional? To what other quantities can we make the interest proportional by keeping certain quantities constant?

34. What is the number of minute spaces, m, traveled by the minute hand of a clock in 60 minutes? the number, h, traveled by the hour hand? What is the relation between m and h? Draw the graph. At 12 o'clock what are the positions of the two hands? Thus, m and h may represent the positions of the two hands m minutes past twelve.

35. What are the positions of the two hands at 3 o'clock? If m and h are to represent the number of minute spaces distance from the figure "XII" of the two hands at m minutes past three, what equation connects m and h? Draw the graph.

Find graphically when $m = h$; that is, when the hour and minute hands will be together. Solve also from the equation.

36. Find graphically and from the equation, when, after 6 o'clock, the hour and minute hands of the clock will be together.

37. Find graphically and from the equation, when, after three o'clock, the hour and minute hands will be exactly opposite one another.

Find also when *before* three o'clock this will happen.

38. What is the total surface area, A, of a closed cylinder whose height is h and the radius of whose base is r? Factor the result. If r is constant, to what variable quantity is A proportional? Assume some convenient value for r, say $r = 7$ or $\frac{7}{2}$ (take $\pi = 3\frac{1}{7}$), and plot the relation between A and h.

Plot on the same diagram the surface area, B, of a corresponding cylinder, open at the top.

Plot on the same diagram the surface area, C, of a cylinder open at both ends.

Compare the three figures.

39. Each of two boxes has a square bottom and rectangular sides. The heights are the same. The first is 2 feet wide and has a closed bottom and top. The second has a bottom, but no top; and it is 3 feet wide. Find the common height of the boxes if the amounts of lumber used in making them are in the ratio of 8 to 13.

PART IV. FRACTIONAL EQUATIONS

81. Fractional Equations. A **fractional equation** is one which contains fractional expressions.

Equations with simple fractional coefficients are not usually included, because they can be so easily solved; but they are none the less fractional equations. Some such equations were solved on p. 58.

Ex. 1. Thus, $\frac{2}{3}x + 5 = 7$ is an easy fractional equation.

Subtract 5 from each side: $\quad \frac{2}{3}x = 2.$

Divide each side by 2: $\quad \frac{1}{3}x = 1.$

Multiply each side by 3: $\quad x = 3.$

Ex. 2. A more typical example is $\dfrac{2x-1}{x+2} = \dfrac{6x}{3x+10}$.

Multiply each side by $x + 2$ and then by $3x + 10$:

$$(3x + 10)(2x - 1) = 6x(x + 2),$$

or,
$$6x^2 + 17x - 10 = 6x^2 + 12x.$$

Subtract $6x^2$ from each side:

$$17x - 10 = 12x.$$

Subtract $12x$ from each side, and add 10 to each side:

$$5x = +10,$$

or,
$$x = 2.$$

This we verify by trying $x = 2$ in the given equation. Thus,

$$\frac{2 \cdot 2 - 1}{2 + 2} = \frac{6 \cdot 2}{3 \cdot 2 + 10}, \text{ or, } \frac{3}{4} = \frac{12}{16},$$

which is true; hence we conclude that $x = 2$ is a correct answer.

As in this example, it is usually best to *clear of fractions* (§ 78) immediately. *To do so, multiply both sides by the L. C. M. of the denominators, if it can be found, or at least by as low a multiple as is practicable.* Or we may *simply cross-multiply, as in* § 78, No. V, p. 138.

Ex. 3. $\dfrac{3}{5x-5}+\dfrac{x-4}{x-1}=\dfrac{2x-6}{2x}$.

The L.C.M. of the denominators is $10x(x-1)$; multiplying both sides by this gives, $2x\cdot 3+10x\cdot(x-4)=5(x-1)(2x-6)$.

(This is also found by cross-multiplying, § 70, No. V.)

Or, $$6x+10x^2-40x=10x^2-40x+30.$$

Subtract $10x^2-40x$ from each side:

$$6x=30, \text{ or, } x=5.$$

Check: $$\frac{3}{5\cdot 5-5}+\frac{5-4}{5-1}=\frac{2\cdot 5-6}{2\cdot 5}, \text{ or, } \frac{3}{20}+\frac{1}{4}=\frac{4}{10}.$$

In case the answer found makes any denominator of the original equation *zero*, that answer must be discarded, since division by zero is impossible. (See § 46, p. 75; § 65, p. 106; § 67, p. 109.)

EXERCISES XI: CHAPTER V

Solve the following equations for the unknown quantities:

1. $\dfrac{y-3}{2y-5}=\dfrac{y}{2y+2}$.

2. $\dfrac{3t-4}{t+2}=\dfrac{6t-11}{2t+1}$.

3. $\dfrac{x-1}{2x+1}=\dfrac{x+3}{2x+12}$.

4. $\dfrac{2t+a}{t-a}=\dfrac{2t+6a}{t+a}$. [Find t.]

5. $\dfrac{3z-5}{2z-15}=\dfrac{6z-25}{4z-33}$.

6. $\dfrac{m^2+4}{2m}=\dfrac{m+1}{2}$.

7. $\dfrac{a^2+a+3}{2a+1}=\dfrac{a+1}{2}$.

8. $\dfrac{10k+35}{2k+8}=\dfrac{5k+5}{k-1}$.

9. $\dfrac{x^2+2x-3}{3x^2+4x+1}=\dfrac{x-1}{x+1}$.

10. $\dfrac{2n+5}{n-2}=\dfrac{6n+20}{3n+4}$.

11. $\dfrac{n-2}{n-4}=\dfrac{n+16}{n+2}$.

12. $\dfrac{x+3}{x+5}=\dfrac{x+6}{x+9}$.

13. $\dfrac{10x-5}{4x-1}=\dfrac{4-5x}{1-2x}$.

14. $\dfrac{k+18}{k+14}=\dfrac{k+3}{k+2}$.

15. $\dfrac{3c-20}{c-2}=\dfrac{3c-12}{c+3}$.

16. $\dfrac{x+1}{x-3}+\dfrac{x-5}{x}=2$.

17. $\dfrac{2x}{x+1}+\dfrac{x+6}{x+3}=3$.

18. $\dfrac{x+2}{x+3}+\dfrac{2}{x+5}=1$.

19. $\dfrac{2p+4}{p-3}+\dfrac{3p-5}{p-10}=5$.

20. $\dfrac{3z+2}{z+5}-\dfrac{2z+14}{2z+1}=2$.

82. Other Cases. If the terms containing x^2 and other higher powers *do not cancel* as they do above, the resulting equations often may be solved nevertheless. (See § 66, p. 106.)

Ex. 1. $\dfrac{2x-3}{x+2}=\dfrac{5x}{3x+10}$.

Clear of fractions:

$$6x^2+11x-30=5x^2+10x.$$

Subtract $5x^2+10x$ from each side:

$$x^2+x-30=0,$$

or, $$(x+6)(x-5)=0.$$

Hence, either $x+6=0$ or $x-5=0$. See § 66.

Whence, $x=-6$ or $x=5$.

Check for $x=5$:

$$\frac{2\cdot 5-3}{5+2}=\frac{5\cdot 5}{3\cdot 5+10}, \text{ or, } \frac{7}{7}=\frac{25}{25} \text{ (check)}.$$

Check for $x=-6$:

$$\frac{2\cdot(-6)-3}{-6+2}=\frac{5(-6)}{3(-6)+10}, \text{ or, } \frac{-15}{-4}=\frac{-30}{-8} \text{ (check)}.$$

Thus there are two answers, either of which is correct.

EXERCISES XII: CHAPTER V

Solve for the unknown quantity:

1. $\dfrac{2x+3}{x+2}=\dfrac{5x}{2x+1}$.

2. $\dfrac{t+3}{t-3}=\dfrac{3t+5}{2t-5}$.

3. $\dfrac{2z+5}{z+1}=\dfrac{z+5}{z+7}$.

4. $\dfrac{n-7}{2n-5}=\dfrac{2n-18}{3n-20}$.

5. $\dfrac{2a-4}{a-3}=\dfrac{3a-6}{a-2}$.

8. $\dfrac{1-x}{x+3}+\dfrac{4x-2}{3x-1}=1$.

6. $\dfrac{b+1}{3b-2}=\dfrac{2b-1}{3b+2}$.

9. $\dfrac{3x+1}{5x}=\dfrac{2x+3}{3x+4}$.

7. $\dfrac{x+3}{2x-5}=\dfrac{3x+2}{x+2}$.

10. $\dfrac{k+1}{k+3}+\dfrac{k-4}{k-1}=1$.

11. $\dfrac{p+1}{p-1}+\dfrac{3p-1}{2p-3}=8$.

12. $\dfrac{p^2-5p+3}{p-9}=\dfrac{2p^2-10p+15}{2p}$.

83. Linear Equations; Other Equations. The equations in §§ 81 and 82 are distinguished by having *only the first power of x in their reduced form after clearing of fractions and simplifying.* Such an equation is called **a simple equation, a linear equation,** or **an equation of the first degree.** (See pp. 25, 58.)

If x^2 is the highest power of x in the reduced form, the equation is called **a quadratic equation** or **an equation of the second degree.**

If x^3 is the highest power of x in the reduced form, the equation is called **a cubic equation,** or **an equation of the third degree.**

If x^4 is the highest power of x in the reduced form, the equation is called **an equation of the fourth degree,** and so on.

In general, if the reduced form contains x^n, but no higher power of x, the equation is called **an equation of the nth degree.**

In these statements it is understood that nothing but simple powers of x multiplied by constant coefficients remain as terms in the *reduced form.*

84. Operations. In this chapter we shall deal principally with equations *of the first degree* or *linear* equations. The operations of importance are:

1. **Clearing of fractions**: explained in § 81, p. 149. (See also § 78, V.)

2. **Multiplying, dividing**, except by zero, **adding, or subtracting by the same number on each side**: explained in §§ 35, 65, pp. 55, 106.

3. **Transposing a term.** *To transpose a term is to carry it from one side of the equation to the other and change its sign;* this amounts to subtracting the term from each side. (Compare § 38, pp. 58–59.)

Thus, if $2x + 4 = 8$, subtracting 4 from each side, we have $2x = 8 - 4$, which amounts to carrying the term 4 to the other side and changing its sign. Similar operations occur often above.

4. **Cancellation of terms.** *To cancel a term that occurs on both sides, remove it* (*or blot it out*) *on each side;* this amounts to subtracting that term from each side.

The student may hereafter use all of these operations. *Care should be taken never to transpose a factor;* it is only *terms* that may be transposed.

The student should beware of *canceling factors in equations*, although this is sometimes a perfectly justifiable process. The danger is that *one may cancel a factor that is equal to zero*, and that would be wrong (§§ 46, 65, pp. 75, 106). For example, although $0 \cdot 5 = 0 \cdot 7$, it does not follow that $5 = 7$. Instead of trying to cancel a factor, carefully use the principle that both sides may be divided by any number except zero. (See IV, § 65, p. 106.)

85. Practical Examples. A few practical examples follow, in which the preceding principles are applied.

Ex. 1. A cistern may be filled by one pipe in 4 hours; by another in 5 hours. It may be emptied by a windmill pump in 6 hours. If the pump and both pipes are running, how long will it take to fill the cistern?

Let x be the number of hours required to fill the cistern.

Then $$\frac{1}{x} = \text{part filled in 1 hour.}$$

The first pipe fills $\frac{1}{4}$ the cistern; the second $\frac{1}{5}$ of it; the pump empties $\frac{1}{6}$ of it, in an hour.

Hence, $$\frac{1}{x} = \frac{1}{4} + \frac{1}{5} - \frac{1}{6},$$

or, $$\frac{1}{x} = \frac{17}{60}.$$

Clear of fractions: $$60 = 17x.$$

Divide by 17: $$x = \tfrac{60}{17} = 3\tfrac{9}{17}.$$

Check: in $\frac{60}{17}$ hours the first pipe fills $\frac{60}{17} \cdot \frac{1}{4} = \frac{15}{17}$ of the cistern; the second fills $\frac{60}{17} \cdot \frac{1}{5} = \frac{12}{17}$ of the cistern; the pump empties $\frac{60}{17} \cdot \frac{1}{6} = \frac{10}{17}$ of the cistern; hence, in $\frac{60}{17}$ hours the cistern has in it $\frac{15}{17} + \frac{12}{17} - \frac{10}{17} = \frac{17}{17}$, *i.e.* it is just full.

Ex. 2. Two teams playing baseball in a league have the following record:

	WON	LOST
A	59	22
B	56	24

These teams play a final series of ten games together. How many must A win in order that the ratio of games won to games lost be greater for A than for B?

Let x be a number A must win to come exactly even with B. Then A would lose $(10 - x)$ of the final games, and the final totals would be: A won $59 + x$, lost $22 + (10 - x)$; B won $56 + (10 - x)$, lost $24 + x$. If the ratio of games won to games lost is the same for both,

$$\frac{59 + x}{22 + 10 - x} = \frac{56 + (10 - x)}{24 + x},$$

or, $$\frac{59 + x}{32 - x} = \frac{66 - x}{24 + x}.$$

Take each of the proportions by "composition" (see I, § 78, p. 137):

$$\frac{91}{32-x}=\frac{90}{24+x}.$$

Clear of fractions: $\quad 91(24+x)=90(32-x),$

or, $\quad 91\cdot 24+91x=90\cdot 32-90x.$

Transpose $91\cdot 24$ and also $-90x$:

$$181x=696,$$

or, $$x=3\tfrac{153}{181}.$$

If it were possible for A to win $3\frac{153}{181}$ games, the two teams would finish equal; as it is, if A wins 3 or less, B will win; if A wins 4 or more, A will win. In baseball leagues the practice is somewhat different from this simple case.

This example illustrates the fact that it is sometimes not necessary to know the answer to an example with any great exactness, for here it is impossible for A to win a *fractional* number of games.

The example also illustrates the occasional usefulness of composition of a proportion, or some other of the processes on p. 138.

EXERCISES XIII: CHAPTER V

1. A cistern is furnished with two pipes, one of which can empty it in 6 hours, the other in 3 hours. How long will it take both, opened simultaneously, to empty the cistern?

2. A can do a certain piece of work in 3 days; B can do it in 4 days; A, B, and C together can do it in $1\frac{1}{3}$ days. How long would it take C alone to do the work?

3. A is 15 years old; B is 25 years old. When will A be $\frac{3}{4}$ as old as B?

4. A is 24 years old; B is 52 years old. When will A's age be half of B's?

5. A steamboat is making 6 miles an hour against the wind on a journey of 30 miles. At a distance of 10 miles after starting the wind ceases. The whole trip occupies 3 hours and 40 minutes. How many miles per hour does the wind retard the boat?

6. A steamboat travels with the wind at the rate of 15 miles an hour, and returns against the wind. The whole trip is 60 miles, and occupies 8 hours. How strong is the wind?

7. What fraction, equal to $\frac{1}{2}$, becomes equal to $\frac{2}{3}$ by the addition of 5 to the numerator and denominator?

8. What number can be added to both terms of the fraction $\frac{5}{9}$, to make a new fraction equal to $\frac{2}{3}$?

9. If the value of a fraction is unchanged by adding 7 to both terms, show that the value of the fraction is unity. (Let $\dfrac{x}{y}$ be the fraction.)

10. If the value of a fraction when 2 is added to both terms is the same as the value when 4 is added to both terms, show that the fraction is equal to unity.

11. What number can be added to 2, 4, 6, and 10 so that the resulting numbers will form a proportion?

12. What must be the dimensions of a rectangle containing 1000 square centimeters, in order that the perimeter may be 266 centimeters?

13. Find the numbers proportional to 1, 2, 3, 4 that may be added regularly to 70, 90, 45, 60, so as to form a proportion.

14. A certain fraction is equal to $\frac{3}{4}$. If from the fraction obtained by adding 10 to both terms the fraction obtained by adding 5 to both terms is subtracted, the difference is $\frac{1}{30}$, what is the fraction? Comment on the two results.

15. The secretary of a club announces that the admission of ten new members would decrease by $1 the assessment required to pay a debt of $20. How many members has the organization, and what is the assessment?

16. A merchant sells to three successive customers one third, one fourth, and one fifth, of a bolt of cloth; he has left 2 yards more than one sixth the original length. Find the original length and the amount sold to each customer.

SUMMARY OF CHAPTER V: FRACTIONS; COMMON FACTORS; REDUCTION; OPERATIONS; PROPORTION; FRACTIONAL EQUATIONS; pp. 118–156

PART I. COMMON FACTORS; REDUCTION OF FRACTIONS, pp. 118–123.

Removal of Common Factor in Numerator and Denominator:

$\dfrac{C \cdot N}{C \cdot D} = \dfrac{N}{D}$; division of both terms by same number; multiplication of both terms by same number; addition (or subtraction) not allowable. § 68, pp. 118–119.

Reduction to Lowest Terms: remove all common factors. Exercises I. § 69, pp. 119–120.

Highest Common Factors: product of all common factors. Exercises II. § 70, pp. 120–121.

Practical work in Factors: removal of all common factors readily found. Exercises III. § 71, pp. 122–123.

PART II. RULES OF OPERATION, pp. 124–136.

Of three Signs in a Fraction (numerator, denominator, whole fraction), change any two. Exercises IV. § 72, pp. 124–125.

Addition and Subtraction of Fractions: reduction to common denominator, and addition of new numerators; simple problems, — common denominator by inspection. Exercises V. § 73, pp. 125–127.

Least Common Multiple: omit duplicate factors.

Choice of Common Denominator: L. C. M., or last multiple practicable; formal rule for difficult problems. Exercises VI. § 74, pp. 127–131.

Product of Fractions: (product of numerators) ÷ (product of denominators). Exercises VII. § 75, pp. 131–133.

Quotient of two Fractions: multiply by the divisor inverted.

Reciprocal of a Number: 1 divided by that number; of a fraction, the fraction inverted; use in division.

Complex Fractions: equivalence to division; caution to mark main horizontal. Exercises VIII. § 76, pp. 134–136.

PART III. PROPORTION, pp. 137–148.

Proportion: an equality of two fractions. § 77, p. 137.

If $\frac{a}{b} = \frac{c}{d}$, then

I. $\frac{a+b}{b} = \frac{c+d}{d}$ ("Composition");

II. $\frac{a-b}{b} = \frac{c-d}{d}$ ("Division");

III. $\frac{a+b}{a-b} = \frac{c+d}{c-d}$ ("Composition and Division");

IV. $\frac{b}{a} = \frac{d}{c}$ ("Inversion");

V. $ad = bc$ ("Clearing of Fractions");

VI. From $ad = bc$, $\frac{a}{b} = \frac{c}{d}$, also $\frac{a}{c} = \frac{b}{d}$, etc. Exercises IX. § 78, pp. 137–138.

Variables in Proportion: $y = kx$. § 79, pp. 139–140.

Equation $y = kx$*:* straight-line figure.

Examples in Proportion. Exercises X. § 80, pp. 140–148.

PART IV. FRACTIONAL EQUATIONS, pp. 149–156.

Clearing of Fractions: multiply both sides by the L.C.M. Exercises XI and XII. §§ 81–82, pp. 149–152.

Degree of an Equation: degree of the highest power of x after simplification; linear or simple, first degree; quadratic, second degree; etc. § 83, p. 152.

Permissible Operations:

(1) Clearing of fractions;

(2) Multiplying, dividing, etc. (both sides);

(3) Transposition of terms;

(4) Cancellation of terms in equations. § 84, p. 153.

Examples stated in English. Exercises XIII. § 85, pp. 153–156.

CHAPTER VI. SIMULTANEOUS LINEAR EQUATIONS

86. Introduction. Problems in which two distinct statements are made lead to two equations which should both be true. Such pairs of equations are called **simultaneous equations.**

If the equations which are formed are *of the first degree* (see pp. 58, 152), the two taken together are said to form a pair of **simultaneous linear equations.**

Ex. 1. Suppose the sum of two numbers is 45 and their difference is 18. What are the numbers?

Notice that such an example may easily arise in a practical problem.

STATEMENT: Let x and y be the two numbers; then

$$\begin{cases} x + y = 45, \text{ i.e. their sum is 45.} & (1) \\ x - y = 18, \text{ i.e. their difference is 18.} & (2) \end{cases}$$

1. THE FIGURE. The first equation *alone* has many different solutions: 10 *and* 35, 11 *and* 34, 12 *and* 33, $10\frac{1}{2}$ *and* $34\frac{1}{2}$, etc. For this reason the first equation alone is called an **indeterminate** equation. (See also Chapter IX, p. 237.) The corresponding graph is a straight line since the equation is of the first degree (§ 80), AB in the figure.

The second equation is likewise an indeterminate equation of the first degree; its graph is the straight line CD in the figure.

Any point on AB corresponds to a pair of numbers whose sum is 45. *Any point on CD corresponds to a pair of numbers whose difference is* 18.

There are many such pairs; but there is evidently *only one point that lies on both lines*, i.e. *there is only one pair of numbers that gives the sum* 45 *and the difference* 18.

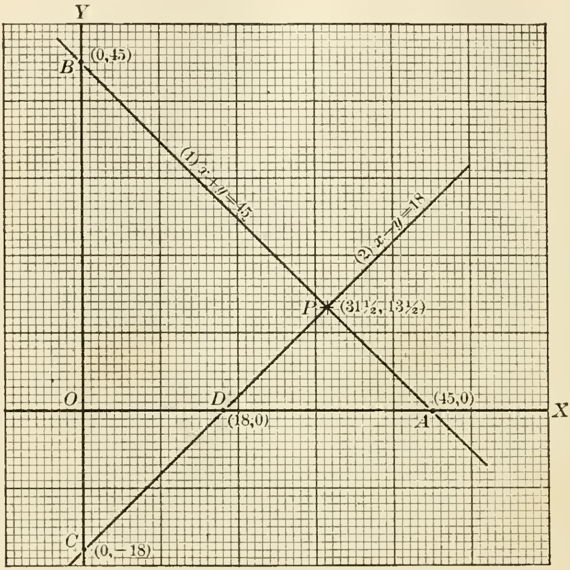


FIG. 23.

A glance at the figure shows that $x = 31\frac{1}{2}$, $y = 13\frac{1}{2}$ are about correct; a *trial* of these numbers shows that they are exactly correct: $31\frac{1}{2} + 13\frac{1}{2} = 45$, $31\frac{1}{2} - 13\frac{1}{2} = 18$.

2. SOLUTION. These answers can be found otherwise by several different methods, one of which follows; other methods are given in § 90.

METHOD I. BY ADDITION OR SUBTRACTION.

Ex. 1.
$$\begin{cases} x + y = 45, & (1) \\ x - y = 18. & (2) \end{cases}$$

We note that the *sums* of these equal numbers are equal; hence, adding the right sides together and the left sides together, we get an equation that contains only the letter x:

$$2x = 63,$$

whence, $x = 31\frac{1}{2}$, which agrees with the figure.

Likewise, subtracting the right sides and also the left sides of the original equations, we get:

$$2y = 27,$$

or, $y = 13\frac{1}{2}$, which agrees with the figure.

Hence, $x = 31\frac{1}{2}$, $y = 13\frac{1}{2}$, are the answers.

Check: $x + y = 31\frac{1}{2} + 13\frac{1}{2} = 45$; $x - y = 31\frac{1}{2} - 13\frac{1}{2} = 18$.

The graph should always be drawn, as on p. 160, as a check on the correctness of the work done, at least until the student is so sure of his ability that there is small chance of error.

Ex. 2. Given
$$\begin{cases} 3x + 2y = 12, & (1) \\ 4x + 5y = 20, & (2) \end{cases}$$
to find x and y.

1. The Figure. To draw the figure:

in (1) $\begin{cases} \text{if } x = 0, y = 6, & (A) \\ \text{if } y = 0, x = 4, & (B) \end{cases}$

in (2) $\begin{cases} \text{if } x = 0, y = 4, & (D) \\ \text{if } y = 0, x = 5. & (C) \end{cases}$

Drawing these points, we find the lines AB and CD to represent equations (1) and (2), respectively. The only common point P gives $x = 1.7$, $y = 2.8$, about. Let us try to solve and find the answers exactly.

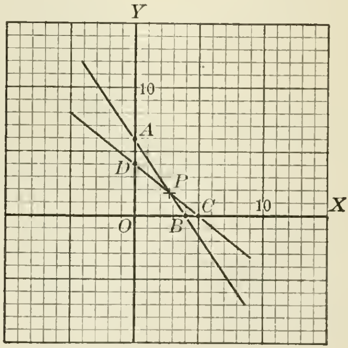


Fig. 24.

Method I. By Addition or Subtraction.

$$\begin{cases} 3x + 2y = 12, & (1) \\ 4x + 5y = 20. & (2) \end{cases}$$

In order to obtain an equation that contains only the letter y, multiply both sides of (1) by 4,

(3) $$12x + 8y = 48,$$

and multiply both sides of (2) by 3,

(4) $$12x + 15y = 60.$$

Subtract (3) from (4):

$$7y = 12,$$
$$y = \tfrac{12}{7} = 1\tfrac{5}{7}.$$

In actual work we write this as follows:

$$\begin{array}{r|l} -4 & 3x + 2y = 12 \\ 3 & 4x + 5y = 20 \\ \hline & \quad\ \ 7y = 12 \\ & \quad\ \ \ \ y = 1\tfrac{5}{7} \end{array}$$

The numbers -4 and 3 indicate the multipliers; notice that -4 is written in place of $+4$, in order that we may *add* instead of subtract.

Likewise, we find an equation that contains only x:

$$\begin{array}{r|l} 5 & 3x + 2y = 12 \\ -2 & 4x + 5y = 20 \\ \hline & 7x \qquad\ = 20 \end{array}$$

whence, $$x = 2\tfrac{6}{7}.$$

Hence, $$x = 2\tfrac{6}{7},\quad y = 1\tfrac{5}{7}.$$

Check: $$3\,(2\tfrac{6}{7}) + 2\,(1\tfrac{5}{7}) = \frac{60 + 24}{7} = \frac{84}{7} = 12.$$

$$4\,(2\tfrac{6}{7}) + 5\,(1\tfrac{5}{7}) = \frac{80 + 60}{7} = \frac{140}{7} = 20.$$

Notice that the answers obtained from the figure are slightly incorrect:

$x = 2\tfrac{6}{7} = 2.8555 \cdots$ (we found 2.8 in the figure).

$y = 1\tfrac{5}{7} = 1.714 \cdots$ (we found 1.7 in the figure).

We might also find x after finding y by putting the value of y found ($y = 1\tfrac{5}{7}$) in place of x in either of the given equations; thus, putting $y = 1\tfrac{5}{7}$ in (1), we find $3x + 2\,(1\tfrac{5}{7}) = 12$, whence $x = 2\tfrac{6}{7}$. But it is better to do the work as shown, because a mistake in finding y does not then cause an error in the value of x also.

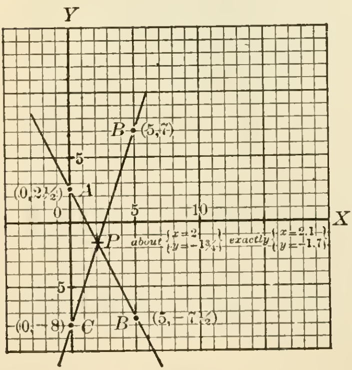


FIG. 25.

Ex. 3. $\begin{cases} y = 3x - 8, \\ 4x + 2y - 5 = 0. \end{cases}$

Transpose the terms to get the equation in the form

$$3x - y = 8,$$
$$4x + 2y = 5.$$

The figure is as shown. The results are therefore *about* $x = 2$, $y = -1\tfrac{3}{4}$.

The solution (Method I, by addition and subtraction) is

$$\begin{array}{r|l} 2 & 3x - y = 8 \\ & 4x + 2y = 5 \\ \hline & 10x = 21 \\ & x = 2.1 \end{array} \qquad \begin{array}{r|l} -4 & 3x - y = 8 \\ 3 & 4x + 2y = 5 \\ \hline & 10y = -17 \\ & y = -1.7 \end{array}$$

Check: $y = 3x - 8$: $-1.7 = 3(2.1) - 8 = 6.3 - 8$ (correct).

$4x + 2y - 5 = 0$: $4(2.1) + 2(-1.7) - 5 = 0$,

or, $8.4 - 3.4 - 5 = 0$ (correct).

Notice that the answers from the figure are not precisely correct; they agree with the precise result as nearly as we could expect.

87. Formal Rule. Elimination. The method of solution just given may be summed up in the following rule:

Multiply both sides of each equation by the coefficient of y [or of x] in the other equation; subtract the two resulting equations.

The new equation has no term in x [or y]; solve it for y [or x]. Proceed similarly to find x [or y].

Any process which, as above, results in getting rid of one of the unknown quantities, is called **elimination**. We say that y (for example) has been *eliminated* when we find a new equation which is free from y. The purpose of eliminating one unknown quantity is to find an equation in one letter alone. Such an equation can be solved by previous rules.

In equations that contain fractions we first clear of fractions, then proceed as above.

EXERCISES I: CHAPTER VI

[In each of the following, draw the graph, and estimate solutions; then solve by addition and subtraction.]

1. The sum of two numbers is 19; their difference is 6. What are the numbers?

2. The sum of two numbers is -10, their difference is 2. What are the numbers?

3. Divide 20 into two parts, one of which shall be four times the other.

4. Divide 35 into two parts, which shall be in the ratio of 3 to 2.

5. What fraction becomes equal to $\frac{2}{3}$ if each term is decreased by 1, and to $\frac{3}{4}$ if each term is increased by 1?

6. Twice the difference of two numbers is 12; twice the greater number exceeds the lesser number by 20. What are the numbers?

7. Half one number exceeds one third another number by unity; the first number is less than the second by unity. What are the numbers?

8. What number of two digits exceeds 7 times the sum of its digits by 3, and exceeds 16 times the difference between the tens' and units' digit by 4?

Solve the following pairs of equations for the letters appearing in them:

9. $\begin{cases} 6p+5q=2, \\ p+3q=9. \end{cases}$

10. $\begin{cases} x+t=7, \\ 3x-10t=8. \end{cases}$

11. $\begin{cases} 2m-n=1, \\ 6n-11m=7. \end{cases}$

12. $\begin{cases} 4x+3y=12, \\ 9x-14y=27. \end{cases}$

13. $\begin{cases} y=5x+3, \\ x=2y-24. \end{cases}$

14. $\begin{cases} 4k-3r=1, \\ 6r-2k=1. \end{cases}$

15. $\begin{cases} 8s+5k+1=0, \\ 4s=10k+7. \end{cases}$

16. $\begin{cases} l-5n=4l-6n, \\ l-(5n-l)+(l+4)=0. \end{cases}$

17. $\begin{cases} -3u+8v=5, \\ u+v=2. \end{cases}$

18. $\begin{cases} (6-x)-(y+3x)=-1, \\ (x+y)+(4x-3y)=12. \end{cases}$

19. $\begin{cases} \frac{1}{2}x+\frac{2}{3}y=3, \\ 2x-y=1. \end{cases}$

20. $\begin{cases} \dfrac{x}{3}-\dfrac{y}{4}=1, \\ \dfrac{5x}{2}-3y=3. \end{cases}$

21. $\begin{cases} \dfrac{s}{3}-\dfrac{t}{5}=1, \\ s-t=1. \end{cases}$

22. $\begin{cases} \dfrac{p}{7}+\dfrac{q}{6}=3, \\ \dfrac{3p}{2}-q=15. \end{cases}$

23. $\begin{cases} \dfrac{z}{3}-\dfrac{t}{4}=\dfrac{1}{12}, \\ \dfrac{z}{2}-\dfrac{3t}{16}=\dfrac{1}{2}. \end{cases}$

Solve the following for the letters indicated; draw the figure for any convenient choice of the other letters:

24. $x + y = a,$
$x - y = b.$ $(x, y.)$

25. $ap - bq = 0,$
$bp + aq = a^2 + b^2.$ $(p, q.)$

26. $ax + by = 2\,xy,$
$bx - ay = x^2 - y^2.$ $(a, b.)$

27. $\frac{m}{x} - \frac{n}{y} = 2,$
$\frac{m}{6\,x} + \frac{n}{2\,y} = 1.$

[SUGGESTION. First solve for $\frac{m}{x}$ and $\frac{n}{y}$ as if for single letters; then solve for m and n.]

28. $a + (n-1)d = l,$
$na + \frac{n(n-1)}{2}d = s.$ $(a, d.)$ (Compare pp. 325, 326, §§ 157, 158.)

29. $(k+1)x + 3\,y = m,$
$mx - 2\,y = k.$ $(x, y;\ k, m.)$

30. $ax + by = mr,$
$bx - ay = nr.$ $(x, y;\ a, b;\ x, r;\ b, r.)$

88. Impossible Case. If a problem contains two statements that are contradictory, the simultaneous equations formed cannot be solved.

Ex. 1. Find the dimensions of a rectangle whose perimeter (the entire boundary) is 10 and whose two dimensions have a sum 7.

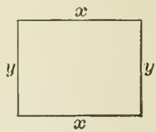


FIG. 26.

Let us try to draw the figure; let x and y be the sides. Then the perimeter is $2\,x + 2\,y$, and

(1) $$2\,x + 2\,y = 10.$$

But

(2) $$x + y = 7,$$

since the sum of the two sides is to be 7.

These two statements are contradictory for, from the first equation, when both sides are divided by 2, $x + y = 5$.

Now $x + y$ cannot be both 5 and 7, hence there is no answer.

We could, however, find many different pairs of numbers that satisfy only one of the two given equations. In fact, (1) by itself is an indeterminate equation of the first degree; it therefore has a straight-line graph, the lower line in Fig. 27.

Likewise, (2) is represented by the upper straight line. Now these lines never cross; for if they did, their common point would be on both, *i.e.* there would be a pair of numbers whose sum is 5 and whose sum is also 7, which is absurd.

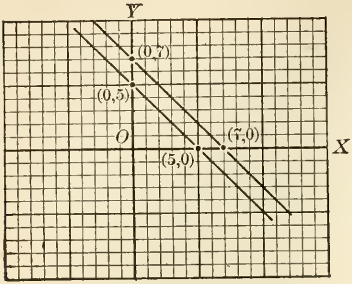


Fig. 27.

Note 1. The kind of argument just used is often called *reductio ad absurdum*, or *reduction to an absurdity;* we prove that the statement made (in this case the statement that (1) and (2) do not cross) is *true* by showing that otherwise an *absurd* (incorrect) conclusion would follow.

Note 2. Straight lines in the same plane that never cross are called **parallel** lines.

Two simultaneous equations in two unknown quantities that correspond to parallel lines in the figure have no pair of solutions.

A pair of numbers that are the solutions of a pair of simultaneous equations corresponds to a point on both lines.

Ex. 2. $\begin{cases} 2y = 6x - 5, & (1) \\ 9x - 3y = -10. & (2) \end{cases}$

Here it is not easy to see by mere inspection that the equations are not solvable. But if we *solve for y in each one*, we find

$$\begin{cases} y = 3x - \frac{5}{2}, & (4), \text{ from } (1) \\ y = 3x + \frac{10}{3}, & (5), \text{ from } (2) \end{cases}$$

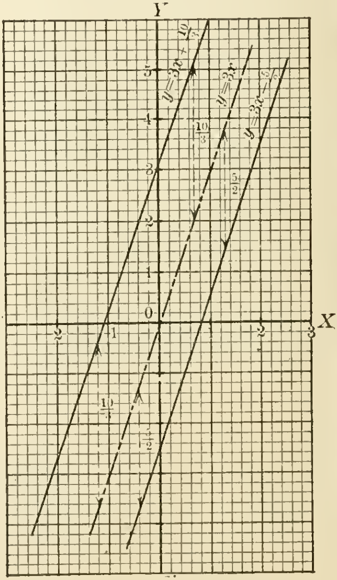


Fig. 28.

These are represented by parallel lines, for each one is parallel to the line.

(6) $y = 3x.$

In fact, (5) is formed by raising (6) vertically by $\frac{10}{3}$, while (4) is formed by lowering (6) vertically by $\frac{5}{2}$. (See p. 25.) It is clear that the three lines are parallel, *i.e.* that they never meet.

Notice that if one does not suspect that a pair of equations has no pair of solutions, attention will be called to the suspicious character by a figure. If a figure shows the lines fairly parallel, we work as above.

The line $y = kx + c$ is parallel to the line $y = kx$; hence two lines are parallel if the coefficient of x is the same in each after solving for y in each.

An attempt to solve two such equations will call attention to the peculiar nature of the example.

Ex. 3.
$$\begin{cases} 2y = 6x - 5, & (1) \\ 9x - 3y = -10. & (2) \end{cases}$$

We should transpose terms to get them in the form below; the solution would then be

$$\begin{array}{r|rcl l} -3 & 6x - 2y &=& 5 & \text{(3), from (1)} \\ 2 & 9x - 3y &=& -10 & \text{(4), the same as (2)} \\ \hline & 0 + \quad 0 &=& -35 & \end{array}$$

This is absurd; -35 cannot be equal to zero. If we did not suspect that there was no solution before, such an absurd result should make us carefully investigate, as above.

Notice that elimination of either letter also eliminates the other.

EXERCISES II: CHAPTER VI

In each of the following plot the equations; solve each equation for one of the unknown letters and compare results:

1. $\begin{cases} 3y = 6x + 2, \\ 4x - 2y = 3. \end{cases}$

2. $\begin{cases} 2x - y = 7, \\ 3x - y = 11 - \dfrac{3 - y}{2}. \end{cases}$

3. $\begin{cases} p - 2q = 7, \\ 5p - q = 3(5 + 3q). \end{cases}$

4. $\begin{cases} 3n + 4r = 6, \\ (2r - n) - 2(n + 3r) + 12 = 0. \end{cases}$

5. Find two integers whose sum is 12, such that the sum of the integers next following them is 18.

6. What fraction becomes equal to $\frac{5}{6}$ if its terms are either each increased by 5 or each decreased by 2?

89. Equivalent Equations. A problem may lead to two equations which are essentially the same one.

Ex. 1.
$$\begin{cases} 2y = 6x - 4, & (1) \\ 9x - 3y - 6 = 0. & (2) \end{cases}$$

Solving for y in each, we get

$$\begin{cases} y = 3x - 2, & (3), \text{ from } (1) \\ y = 3x - 2. & (4), \text{ from } (2) \end{cases}$$

In such a case any pair of numbers that satisfies one equation also satisfies the other. The equations are called equivalent, for one may be reduced to the other by operations which are correct.

The figure in this case is the same for both equations; it is a straight line, since each equation is of the first degree.

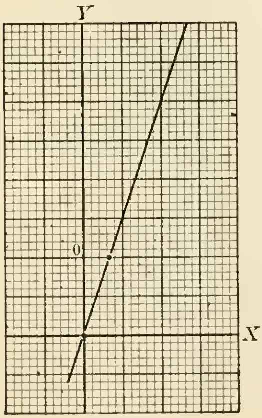


FIG. 29.

There are, of course, *an indefinitely large number of pairs of numbers that satisfy both equations*, for any point on the straight line gives such a pair.

To test for this kind of problem, we solve for y in each. If the result is precisely the same in each case, we know the equations are equivalent. Attention will be called to such a case (just as in § 88) by drawing a figure, or also by an attempt to solve by the ordinary method.

It is well to notice that if two straight lines have two points in common, they are the same line; *i.e.* if two

equations of the first degree have two pairs of solutions, they are equivalent equations, and therefore have an indefinitely large number of pairs of solutions.

Two simultaneous equations of the first degree have

$$\left\{\begin{matrix}\text{one solution}\\ \text{no solution}\\ \text{many solutions}\end{matrix}\right\} \text{ if the two lines are } \left\{\begin{matrix}\text{intersecting lines}\\ \text{parallel lines}\\ \text{the same line}\end{matrix}\right\}.$$

EXERCISES III: CHAPTER VI

In each of the following, plot the equations; solve each equation for one of the unknowns and compare the results:

1. $\begin{cases} 3x - 5y = 1, \\ 10y = 6x - 2. \end{cases}$

2. $\begin{cases} \dfrac{p}{3} = \dfrac{q}{2} + 5, \\ 4p - 6q = 60. \end{cases}$

3. $\begin{cases} 9l + 2 = 2(5 - m), \\ 4l + m = \dfrac{8 - l}{2}. \end{cases}$

4. $\begin{cases} ax - by = a^2 + b^2, \\ \dfrac{x - a}{b} - \dfrac{y + b}{a} = 0. \end{cases}$ (Solve for x, y.)

5. $\begin{cases} x = 10y + 3, \\ (3x - 25y) - 5(y + 2) = -1. \end{cases}$

6. Twice the sum of two numbers is 12; the difference between the numbers lacks twice the smaller number of being equal to 6. What are the numbers?

7. A number of two digits is equal to four times the sum of its digits; if the digits are reversed, the new number is equal to seven times the sum of the digits. What is the number?

8. What fraction becomes equal to $\frac{2}{3}$ if its terms are each diminished by 1; or also if its denominator is diminished by 7 and its numerator by 5?

90. Solution by Substitution. The problems given above may be solved by other methods; one of these is the method called **solution by substitution.** In this method, also, we first *eliminate* one of the unknown letters by a somewhat different process illustrated in the following examples, and then solve for the one which remains, as before.

Ex. 1. (Ex. 1, p. 160.) $\begin{cases} x+y=45, & (1) \\ x-y=18. & (2) \end{cases}$

Solve (1) for y:

(3) $$y = 45 - x.$$

Substitute the value found $(45 - x)$ in the place of y in (2):

$$x - (45 - x) = 18,$$

or, $$2x - 45 = 18.$$

Solving for x, we have $x = 31\tfrac{1}{2}$.

Put this value of x in place of x in (3):

$$y = 45 - 31\tfrac{1}{2} = 13\tfrac{1}{2}.$$

These are the answers found above (p. 161).

Check: $x + y = 31\tfrac{1}{2} + 13\tfrac{1}{2} = 45$ (correct).

$x - y = 31\tfrac{1}{2} - 13\tfrac{1}{2} = 18$ (correct).

Ex. 2. (Ex. 2, p. 161.) $\begin{cases} 3x+2y=12, & (1) \\ 4x+5y=20. & (2) \end{cases}$

Solve (1) for y:

(3) $$y = \frac{12 - 3x}{2}.$$

Substitute for y in (2):

$$4x + 5\left(\frac{12 - 3x}{2}\right) = 20.$$

Clear of fractions: $8x + 60 - 15x = 40.$

Simplify: $7x = 20.$

Solve for x: $x = 2\tfrac{6}{7}.$

Substitute $2\tfrac{6}{7}$ for x in (3):

$$y = \frac{12 - 3(2\tfrac{6}{7})}{2} = \frac{\frac{84}{7} - \frac{60}{7}}{2} = \frac{24}{14} = \frac{12}{7} = 1\tfrac{5}{7}.$$

These are the answers found above (p. 162).

Check: $3x + 2y = 3(\frac{20}{7}) + 2(\frac{12}{7}) = \dfrac{60 + 24}{7} = 12$ (correct).

$$4x + 5y = 4(\tfrac{20}{7}) + 5(\tfrac{12}{7}) = \frac{80 + 60}{7} = 20 \text{ (correct).}$$

Ex. 3. (Ex. 3, p. 162.) $\begin{cases} y = 3x - 8, & (1) \\ 4x + 2y - 5 = 0. & (2) \end{cases}$

Substitute $(3x - 8)$ for y in (2):

(3) $$4x + 2(3x - 8) - 5 = 0,$$

or, $$10x - 16 - 5 = 0,$$

or, $$10x = 21,$$

or, $$x = 2.1.$$

Substitute 2.1 for x in (1):

$$y = 3(2.1) - 8 = 6.3 - 8 = -1.7.$$

These are the answers found above (p. 163).

Check: $y = 3x - 8;\ -1.7 = 3(2.1) - 8$ (correct).

$4x + 2y - 5 = 0;\ 4(2.1) + 2(-1.7) - 5 = 8.4 - 3.4 - 5 = 0$ (correct).

This last example is easier by this method than by the previous method.

EXERCISES IV: CHAPTER VI

Solve the following by substitution; in each case draw the figure:

1. $\begin{cases} x + y = 17, \\ 3x - 5y = 11. \end{cases}$ **2.** $\begin{cases} 2a - 5b = 9, \\ a + 3b = -1. \end{cases}$ **3.** $\begin{cases} 7y + 9u = 8, \\ 5u - 3y = 1. \end{cases}$

4. $\begin{cases} 6x + 9t = 54, \\ 5x - 3t = 3. \end{cases}$

5. $\begin{cases} an + bm = 0, \\ am - bn = l(m^2 + n^2). \end{cases}$ (Solve for a, b.)

6. $\begin{cases} 3x + 4y = 5, \\ 8x + 9y = 10. \end{cases}$

7. $\begin{cases} \dfrac{x}{5} + \dfrac{3y}{7} = 9, \\ \dfrac{2x}{5} + \dfrac{y}{14} = 7. \end{cases}$

8. $\begin{cases} \dfrac{x}{6} + \dfrac{y}{4} = \dfrac{3}{2}, \\ \dfrac{x}{4} - \dfrac{y}{6} = \dfrac{1}{12}. \end{cases}$

91. Solution by Comparison. A third method of solution, called *solution by comparison*, is illustrated by the following examples. This is also a process of *elimination*.

Ex. 1.
$$\begin{cases} x+y=45, & (1)\\ x-y=18. & (2)\end{cases}$$

Solve each equation for y:

$$\begin{cases} y=45-x, \text{ from (1)}, & (3)\\ y=x-18, \text{ from (2)}. & (4)\end{cases}$$

Since $45-x$ and $x-18$ are each equal to y, they are themselves equal, for they are *the same number as* y; hence,

$$45-x=x-18.$$

Solve for x: $\qquad x=31\frac{1}{2}$.

Substitute $31\frac{1}{2}$ for x in (3): $\qquad y=45-31\frac{1}{2}=13\frac{1}{2}$.

These are the answers found above (p. 161).

Check: $\qquad x+y=31\frac{1}{2}+13\frac{1}{2}=45$ (correct).

$\qquad x-y=31\frac{1}{2}-13\frac{1}{2}=18$ (correct).

Ex. 2.
$$\begin{cases} 3x+2y=12, & (1)\\ 4x+5y=20. & (2)\end{cases}$$

Solve for x in each:

$$\begin{cases} x=\dfrac{12-2y}{3}, & (3)\\[2ex] x=\dfrac{20-5y}{4}. & (4)\end{cases}$$

Hence, $$\frac{12-2y}{3}=\frac{20-5y}{4}.$$

Solving this fractional equation for y, we find

$$y=\tfrac{12}{7}=1\tfrac{5}{7}.$$

Substituting $\frac{12}{7}$ for y in (3), we find

$$x=\frac{12-2(\frac{12}{7})}{3}=\frac{\frac{84}{7}-\frac{24}{7}}{3}=\frac{60}{21}=\frac{20}{7}=2\tfrac{6}{7}.$$

Hence, $x=2\frac{6}{7}$, $y=1\frac{5}{7}$, which are the answers found above (p. 162).

EXERCISES V: CHAPTER VI

Solve the following by comparison. Draw the graphs:

1. $\begin{cases} x-y=3, \\ 3x+5y=25. \end{cases}$

2. $\begin{cases} 2m-3n=2, \\ 5m+n=56. \end{cases}$

3. $\begin{cases} p-aq=k, \\ ap+q=l. \end{cases}$ (For p, q.)

4. $\begin{cases} 6r-s+1=0, \\ 10r-3s+11=0. \end{cases}$

5. $\begin{cases} 4x+y=1, \\ 2x+3y=2. \end{cases}$

6. $\begin{cases} \dfrac{x}{3}+\dfrac{y}{5}=4, \\ \dfrac{x}{2}-\dfrac{y}{10}=2. \end{cases}$

92. Three Unknowns. Problems in which three unknowns occur can be solved similarly. We shall illustrate by an example the method of (1) *addition* or *subtraction*, which is most generally used.

$$\text{Ex. 1.}\quad \begin{cases} 3x+y-2z=11, & (1) \\ 2x+3y+z=19, & (2) \\ x+3y-2z=1. & (3) \end{cases}$$

The solution by addition or subtraction is as follows:

$$\begin{array}{r|rcl l} 1 & 3x+\ \ y-2z &=& 11 & (1) \\ 2 & 2x+3y+\ \ z &=& 19 & (2) \\ \hline & 7x+7y\quad\ \ &=& 49 & \\ \text{or,} & x+\ \ y\quad\ \ &=& 7 & \end{array} \qquad \begin{array}{r|rcl l} 1 & 3x+\ \ y-2z &=& 11 & (1) \\ -1 & x+3y-2z &=& 1 & (3) \\ \hline & 2x-2y\quad\ \ &=& 10 & \\ \text{or,} & x-\ \ y\quad\ \ &=& 5 & \end{array}$$

Taking the resulting two equations, we have

$$\begin{array}{rcl} x+y &=& 7 \\ x-y &=& 5 \\ \hline 2x &=& 12 \\ x &=& 6 \end{array} \qquad \begin{array}{r|rcl} 1 & x+y &=& 7 \\ -1 & x-y &=& 5 \\ \hline & 2y &=& 2 \\ & y &=& 1 \end{array}$$

or,

Substituting 6 for x and 1 for y in (1), we have

$$18+1-2z=11, \text{ or } z=4.$$

Check: $3x+y-2z=18+1-8=11$ (correct).
$2x+3y+z=12+3+4=19$ (correct).
$x+3y-2z=6+3-8=1$ (correct).

In this work we may group equations in pairs in any other way we please. Thus, we may take (1) with (2) and eliminate y; (2) with

(3) and eliminate y; the two resulting equations contain only x and z, for which we can solve. In any case the purpose is to reduce this problem to one that we are already able to solve, by eliminating one letter, thus leaving two equations in two letters, which we solve by the preceding methods. The work when y is eliminated follows:

$$\begin{array}{r|l}
3 & 3x + y - 2z = 11 \quad (1) \\
-1 & x + 3y - 2z = 1 \quad (3) \\
\hline
 & 8x \qquad - 4z = 32
\end{array}
\qquad
\begin{array}{r|l}
1 & 2x + 3y + z = 19 \quad (2) \\
-1 & x + 3y - 2z = 1 \quad (3) \\
\hline
 & x \qquad + 3z = 18
\end{array}$$

or, $$2x - z = 8$$

$$\begin{array}{r|l}
3 & 2x - z = 8 \\
1 & x + 3z = 18 \\
\hline
 & 7x \qquad = 42 \\
 & x \qquad = 6
\end{array}
\qquad
\begin{array}{r|l}
-1 & 2x - z = 8 \\
2 & x + 3z = 18 \\
\hline
 & 7z = 28 \\
 & z = 4
\end{array}$$

Hence, by (1) $3 \cdot 6 + y - 2 \cdot 4 = 11$, or $y = 1$.

These are the answers just found.

Let the student solve the same problem by pairing (1) and (3) and (2) and (3), eliminating x in each pair, and solving the resulting pair for y and z.

Equations in more than three letters are solved upon the same plan; such equations will rarely occur in practical problems on topics now known to the student, but they sometimes come up in more advanced work.

EXERCISES VI: CHAPTER VI

Solve the following:

1. $\begin{cases} 2x - 3y + 5z = 15, \\ x + 2y - z = 4, \\ 5x - y + 3z = 19. \end{cases}$

2 $\begin{cases} 3a + b - 7c = 9, \\ 6a - 2b - c = 9, \\ 3a + 4b - 10c = 9. \end{cases}$

3. $\begin{cases} x - y + t = 8, \\ 2x + 5y + 3t = 11, \\ 6x + 11y - 5t = 9. \end{cases}$

4. $\begin{cases} 3x + y - z = 5, \\ 2x - y + 3z = 13, \\ x + 2y - 2z = 0. \end{cases}$

5. $\begin{cases} y + z = 8, \\ z + x = 12, \\ x + y = 10. \end{cases}$

6. $\begin{cases} y + z - x = 1, \\ z + x - y = 3, \\ x + y - z = 9. \end{cases}$

7. $\begin{cases} p + q + r = 7, \\ p + 2q + 3r = 10, \\ 2p + 3q + 6r = 15. \end{cases}$ √

8. $\begin{cases} x + 2y + 3z + t = 11, \\ 5x + 3y - 2z + 2t = 5, \\ -x + y + 4z - t = 6, \\ 2x + 4y - z + 3t = 11. \end{cases}$ **9.** $\begin{cases} q + r + s = 4, \\ r + s + p = 6, \\ s + p + q = 3, \\ p + q + r = 2. \end{cases}$

10. $\begin{cases} y + z - 3x = a - b - c, \\ z + x - 3y = b - c - a, \\ x + y - 3z = c - a - b. \end{cases}$ [Solve for x, y, z; also for a, b, c.]

93. Practical problems are given below which lead to simultaneous equations. No new principle is involved.

EXERCISES VII: CHAPTER VI

Solve the following problems by the use of two or more unknown quantities:

1. A vessel goes downstream at the rate of 6 miles an hour, and upstream at the rate of 4 miles an hour. What would be the rate of the vessel in still water, and what is the rate of the current?

Let s = rate of the vessel in still water, in miles per hour.
c = rate of the current, in miles per hour.

Then, $s + c = 6,\ s - c = 4.$
Hence, on solving, $s = 5,\ c = 1.$

The vessel would make 5 miles per hour in still water, and the current flows at the rate of 1 mile per hour. These results evidently check.

2. A boat can be rowed by its crew 12 miles an hour with the current; but only 3 miles an hour against the current. Find the speed of the current, and the rate at which the boat can be rowed in still water.

3. A boat is rowed for 3 hours with the current, and then rowed back for 3 hours, the total distance thus covered being 27 miles; it is then allowed to drift for an hour, and rowed with the current another hour, thus covering 9 miles. What is the rate of the boat in still water, and how swift is the current?

4. A man has 14 coins, all dollars and quarters, amounting to \$8. How many of each denomination has he?

5. Ten bills of denominations \$2 and \$5 amount to \$29. How many of each denomination are there?

6. I want the equation $ax + by = 1$ to be satisfied by $x = 1$, $y = 1$, and also by $x = -3$, $y = -4$. How must a and b be chosen?

If the equation is to be satisfied by $x = 1,\ y = 1,$

then, $a + b = 1,$

and if by $x = -3$, $y = -4$, then, $-3a - 4b = 1.$

Solving, we have $a = 5,\ b = -4.$

The equation is $5x - 4y = 1.$

Check:

$$5(1) - 4(1) = 5 - 4 = 1.$$
$$5(-3) - 4(-4) = -15 + 16 = 1.$$

Determine l and b, and write the exact form of the equation $y = lx + b$ if it is to be satisfied by:

7. $x = 2,\ y = 11$; and $x = -1,\ y = 2$.

8. $x = 1,\ y = 5$; and $x = 2,\ y = 3$.

9. $x = 0,\ y = 0$; and $x = m,\ y = n$.

Determine a and b, and simplify the equation $ax^2 + by^2 = 400$ if it is to be satisfied by:

10. $x = 12,\ y = 16$; and $x = -16,\ y = 12$.

11. $x = 3,\ y = 5$; and $x = 1,\ y = -1$.

FIG. 30.

If two weights, w and W, balance when placed on opposite sides of a lever at distances d and D, respectively, from the single intermediate point (known as the fulcrum) on which the lever rests, then it is known that $dw = DW$.

12. Weights of 12 and 18 pounds are fastened to the ends of a ten-foot pole. Where must the pole be supported in order that the weights may balance?

13. An unknown weight 3 inches from the fulcrum of a lever is balanced by another unknown weight 9 inches from the fulcrum; an addition of 2 pounds to the first weight necessitates the removal of the second weight 3 inches farther from the fulcrum in order to preserve the balance. What are the two weights?

14. Two unknown weights balance when placed 6 and 9 inches from the fulcrum of a lever; if their positions are reversed, 2 pounds must be added to the lesser weight to restore the balance. What are the weights?

15. An 8-pound weight is placed at an unknown distance from the fulcrum of a lever; on the opposite side is placed an unknown weight. In order to make the lever balance we may either increase the unknown weight by 1 pound and station it 3 inches from the fulcrum, or else station it 4 inches from the fulcrum and add 2 pounds to the other weight. Find the unknown weight and the distance from the fulcrum to the other weight.

16. The sum of the three angles of a triangle is 180°. What are the acute angles of a right-angled triangle if one is twice the other?

17. Determine the three angles of a triangle if the sum of the first and second is twice the third, and the sum of the first and third is the second.

18. The equation $x^2 + y^2 = ax + by + c$ is to be satisfied by $x = 4$, $y = 2$; $x = 4$, $y = -4$; and $x = 5$, $y = 1$. Determine a, b, and c.

19. A boatman in 2 hours rows a certain distance up a stream where the rate of the current is known to be 2 miles an hour; he then rows back to a place 1 mile beyond the starting point in 1 hour. Find the distance he rows each way, and the rate of the boat in still water.

REVIEW EXERCISES VIII: CHAPTER VI

Solve the following sets of equations for the unknown quantities:

1. $\begin{cases} 7x + 5y = 3, \\ 2x + 3y = 4. \end{cases}$

2. $\begin{cases} 5a - 7b - c = 16, \\ 3a = 2b + 2c = 10, \\ 2a + b + 3c = 6. \end{cases}$

3. $\begin{cases} x^2 + y^2 = 13, \\ x^2 - y^2 = 5. \end{cases}$

[Suggestion. First solve for x^2 and y^2; then find x and y.]

4. $\begin{cases} 5k^2 + 3r^2 = 32, \\ 2r^2 - 3k^2 = 15. \end{cases}$

5. $\begin{cases} 7(p+q) - 8(p-q) = 23, \\ 4(p+q) + (p-q) = 41. \end{cases}$

6. $\begin{cases} (c+a) + (a+b) = 1, \\ (a+b) + (b+c) = 0, \\ (b+c) + (c+a) = -1. \end{cases}$

7. $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = 6, \\ \dfrac{1}{x} - \dfrac{1}{y} = 4. \end{cases}$

[Suggestion. First solve for $\dfrac{1}{x}$ and $\dfrac{1}{y}$; then find x and y.]

8. $\begin{cases} \dfrac{1}{v} + \dfrac{1}{w} = \dfrac{5}{6}, \\ \dfrac{1}{w} + \dfrac{1}{u} = \dfrac{7}{12}, \\ \dfrac{1}{n} + \dfrac{1}{v} = \dfrac{3}{4}. \end{cases}$

[Suggestion. First solve for $\dfrac{1}{u}, \dfrac{1}{v}, \dfrac{1}{w}$; then find u, v, w.]

9. $\begin{cases} \dfrac{4}{5-x} + \dfrac{8}{y+1} = 3, \\ \dfrac{5}{5-x} + \dfrac{4}{y+1} = 3. \end{cases}$

[Suggestion. First solve for $\dfrac{1}{5-x}$ and $\dfrac{1}{y+1}$.]

10. $\begin{cases} \dfrac{a}{x} + \dfrac{b}{y} = p, \\ \dfrac{b}{x} - \dfrac{a}{y} = q. \end{cases}$

[Solve for a, b; also for x, y.]

11. $\begin{cases} \dfrac{1}{z+x} + \dfrac{1}{x+y} = 8, \\ \dfrac{1}{x+y} + \dfrac{1}{y+z} = 7, \\ \dfrac{1}{y+z} + \dfrac{1}{z+x} = 7. \end{cases}$

[Suggestion. First solve for $\dfrac{1}{x+y}$, $\dfrac{1}{x+z}$, and $\dfrac{1}{y+z}$; then find $x+y$, $x+z$, and $y+z$; then solve for x, y, z.]

12. $\begin{cases} \dfrac{3}{x-y} + \dfrac{4}{x+y} = 5, \\ \dfrac{15}{x-y} - \dfrac{2}{x+y} = 3. \end{cases}$

13. $\begin{cases} \dfrac{1}{x^2} - \dfrac{5}{y^2} = 5, \\ \dfrac{2}{5x^2} + \dfrac{3}{y^2} = 22. \end{cases}$

14. $\begin{cases} \dfrac{2}{x^2+y^2}+\dfrac{3}{3x^2-2y^2}=\dfrac{9}{50}, \\ \dfrac{5}{x^2+y^2}+\dfrac{6}{3x^2-2y^2}=\dfrac{2}{5}. \end{cases}$

15. $\begin{cases} \dfrac{1}{m-n}+\dfrac{1}{m+n}=a, \\ \dfrac{1}{m-n}-\dfrac{1}{m+n}=b. \end{cases}$

[Solve for m, n.]

16. A and B can do a piece of work in 2 days; but an equal piece of work when A puts in only half his time and B only one third his time requires $4\frac{1}{2}$ days. How long would the work take A and B each, working alone?

17. Two pipes empty a reservoir in 1 hour and 12 minutes. When one pipe is used to fill and one to empty the reservoir, it becomes full in 3 hours. How long would it take each pipe separately to empty the reservoir?

18. A boat can be rowed downstream 4 miles in the same time as it can be rowed upstream 3 miles. A trip 6 miles downstream and back requires $3\frac{1}{2}$ hours. Find the rate of the boat in still water, and the rate of the current.

19. An investment yielding simple interest amounts to \$2080 in 4 years, and to \$2210 in 6 years. Find the amount of the investment and the rate of interest.

20. A, B, and C are at various times engaged to do certain equal pieces of work. B and C together complete the work in 2 hours; C and A in $1\frac{1}{2}$ hours; A and B in $1\frac{1}{5}$ hours. In what time can each do the work alone, and in what time can all three do the work?

SUMMARY OF CHAPTER VI: SIMULTANEOUS LINEAR EQUATIONS, pp. 159–180

Figure: two intersecting straight lines.

Answers: pair of values at point of intersection.

Algebraic Solution, Method I: remove one letter by addition or subtraction.

Elimination: any process for removing one letter.
§ 86, pp. 159–163.

Formal Rule, Method I, by Addition or Subtraction: essentially, multiplication of both sides of each equation by the coefficient of one letter in the other; Exercises I. § 87, pp. 163–165.

Impossible Case: figure, parallel lines; no answers; contradictory conditions, revealed by attempt to solve; Exercises II.
§ 88, pp. 165–168.

Equivalent Equations: figure, coincident lines; answers, one pair of solutions given by any point on the line.

Classification of Equations: Correspondence of $\left\{\begin{array}{c}\text{intersecting}\\ \text{parallel}\\ \text{coincident}\end{array}\right\}$ lines to $\left\{\begin{array}{c}\text{one solution}\\ \text{no solution}\\ \text{many solutions}\end{array}\right\}$; Exercises III. § 89, pp. 168–169.

Solution by Substitution (Method II): essentially, solve one equation for one letter and substitute this value in the other; Exercises IV. § 90, pp. 170–171.

Solution by Comparison (Method III): essentially, solve each equation for one of the letters and equate the values found; Exercises V. § 91, pp. 172–173.

Three Unknowns: elimination of one letter, reduction of problem to two equations in two unknowns; Exercises VI.
§ 92, pp. 173–175.

English Problems: solution of typical examples; Exercises VII.
§ 93, pp. 175–177.

Review Exercises for Chapter VI: Exercises VIII. pp. 178–179.

CHAPTER VII. SIMPLE POWERS AND ROOTS

PART I. POWERS AND ROOTS OF NUMBERS

94. Introduction. The student is already familiar with ordinary powers and roots (§§ 9–10, pp. 8–9).

By definition, $x^n = x \cdot x \cdot x \cdots x$ (n times) is called the **nth power** of a. (See p. 8.)

The process of raising a number to a power is often called **involution.**

By definition, if $b = a \cdot a \cdot a \cdots a$ (n times), then $a = \sqrt[n]{b}$, *i.e.* the **nth root** of b. (See p. 9.)

The process of extracting a root is often called **evolution.**

[Let the student again state the definitions given in § 9, pp. 8–9.]

There are often two answers for a *root* of a number. Thus, $4 = 2 \cdot 2$; hence, 2 is one answer for the *square root* of 4. But $4 = (-2) \cdot (-2)$ also; hence, -2 is also an answer for the square root of 4. When no sign is written before a $\sqrt{\ }$ we shall understand that the *positive* answer is intended, and *we shall agree to distinguish between the two roots by writing, for example:*

$$+2 = \sqrt{4}, \qquad -2 = -\sqrt{4},$$

the negative answer in any case being said to be the *negative* square root. Similarly, $\sqrt{a^2} = +a$, although $-a$ is also one square root of a^2.

There is only one cube root of any number among numbers we know at present. Thus, $8 = 2 \cdot 2 \cdot 2$, hence $2 = \sqrt[3]{8}$, but -2 does not equal $\sqrt[3]{8}$, for $(-2)(-2)(-2) = -8$. In fact, $-2 = \sqrt[3]{-8}$.

A negative number has no square root among numbers we now know; for *the square of any number* we know at present *is positive*, except zero, whose square is zero.

In fact, since an even number of either + or − factors gives + in the product, an *even root* of a negative quantity is meaningless, at present, to the student.

Such roots, even roots of negative quantities, are called **imaginary**. Later we shall discuss them (Appendix, § 31), and we find it possible to use them sometimes. Just now we shall have nothing to do with them. In this chapter, in even roots, we shall assume that the letters used have positive values.

If any *even* root (*i.e.* square root, fourth root, etc.) of any number is known, then the negative of the known answer is also an answer, as above.

On the other hand, the negative of a known answer for an *odd* root (*i.e.* cube root, fifth root, etc.) is *not* an answer, for the odd number of negative signs produce −, not +, in the answer. There is only one answer (among numbers we know at present) for any odd root of a number, since an increase or decrease in the answer would correspond to a larger or a smaller number than the given number.

95. Figure. We have seen how to indicate the squares of numbers in a figure. (See p. 30.) Thus, if $x =$ a given number and $y =$ the square of the given number, we have $y = x^2$. Let us give x various values, as on p. 30, and find values of y. This gives a table as follows:

x	1	−1	2	−2	±3	±4	±5	±6	&c.	$+\frac{1}{2}$	$\pm\frac{1}{4}$	$\pm\frac{3}{4}$	$\pm\frac{1}{3}$	$+\frac{2}{3}$	$\pm\frac{3}{2}$	$\pm\frac{5}{2}$	$\pm\frac{7}{2}$	&c.
y	1	1	4	4	9	16	25			$\frac{1}{4}$	$\frac{1}{16}$	$\frac{9}{16}$	$\frac{1}{9}$	$\frac{4}{9}$	$\frac{9}{4}$	$\frac{25}{4}$		

[Let the student fill in the blank spaces and extend the table, especially by putting in more fractional values.]

Drawing these values in a figure, as on p. 30, we get a graph like Fig. 31. Since this is very narrow for use, we take a longer unit space (one heavy-marked space) horizontally to produce Fig. 32, which represents the same figure on an enlarged scale horizontally (enlarged one way only).

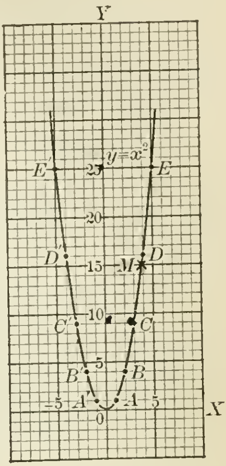


FIG. 31.

To find the square root of a number, we merely look for the given number in the table marked y; its square root is the corresponding x, for we squared each x to get the corresponding y.

Or, in the figure, we count off the given number vertically (on the y line), find the corresponding point on the curve, and then count horizontally to find x.

Thus, if $y=9$, we rise 9 spaces on the y line; the corresponding point of the curve is C; the horizontal space out to C is 3. The point C' is also at a height 9; its x is -3. Hence, the square root of 9 is either 3 or -3.

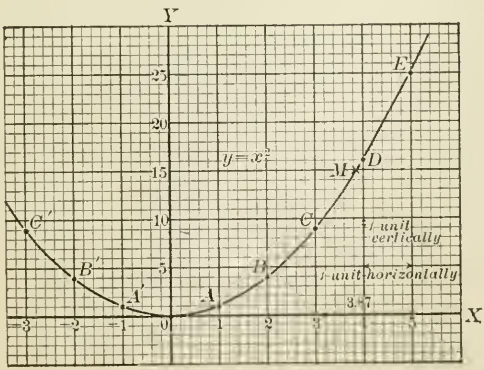


FIG. 32.

96. Inexact Roots. There are numbers that are not in the y table. For example, 15 is not to be found in the y table, no matter how complete the table be made.

We may still carry out the process in the figure. Thus, rising 15 points, we find a point marked M on the curve; counting horizontally out to it, we find $x = 3.8^+$.

In fact, we have: $(3.8)^2 = 14.44$, and $(3.9)^2 = 15.21$.

Hence, $(3.9)^2$ is too large, while $(3.8)^2$ is too small; that is, $x = 3.8$ gives a point $(x = 3.8, y = 14.44)$ *below* M, while $x = 3.9$ gives a point $(x = 3.9, y = 15.21)$ *above* M.

With a more precise figure we could find values of x still closer together — differing in any place of decimals we please — which give points above and below M. Thus, while many numbers do not give exact square roots, we can interpolate the number between two which differ in a small amount. This process gives a result to any number of decimals desired, but it is not exact. The negative of the answer thus found is also an answer, as above.

Any inexact root may be left merely indicated by means of the radical sign; thus $\sqrt{15}$. Any expressed root, whether exact or inexact, is called a **radical**. Any expression which contains a radical is called a **radical expression**. (See also pp. 186, 192, 284.)

97. Formal Process. A process for finding the decimal places is based on the rule $(x + y)^2 = x^2 + 2xy + y^2$, p. 93.

We found $\sqrt{15} = 3.8^+$. Suppose the digit in the next place of decimals is d, then squaring $3.8 + d(.01)$, we find

$$\begin{aligned}\{3.8 + d(.01)\}^2 &= (3.8)^2 + 2d(3.8)(.01) + d^2(.0001)\\ &= (3.8)^2 + d(.076) \pm \text{(very small number)}.\end{aligned}$$

Since we wish to get near to 15, we write

$$(3.8)^2 + d(.076) = 15 \text{ (nearly)},$$

or,
$$d(.076) = 15 - (3.8)^2 \text{ (nearly)},$$

or,
$$d = \frac{15 - (3.8)^2}{.076} \text{ (nearly)}.$$

We can tell by this very closely what d must be; since d is an *integer*, we can guess it *exactly*.

The work is written as follows:

$$
\begin{array}{rr|r l}
 & & 15 & \\
(3.8)^2 = & & 14.44 & \underline{|3.8\,|\,7} \\
\hline
2 \times 3.8 \times .01 = & .0760 & .5600 & \\
(.01)^2 \times 7 = & \underline{.0007} & .5369 & \\
\cline{3-3}
 & .0767 & .0231 &
\end{array}
$$

We continue upon the same plan:

$$
\begin{array}{rr|r l}
2 \times 3.87 \times .001 = & .007740 & .023100 & \underline{|3.87\,|\,2\,|\,9} \\
(.001)^2 \times 2 = & \underline{.000002} & & \\
 & .007742 & .015484 & \\
\hline
2 \times 3.872 \times .0001 = & .00077440 & .00761600 & \\
(.0001)^2 \times 9 = & \underline{.00000009} & & \\
 & .00077449 & .00697041 & \\
\cline{2-3}
 & & .00064559 &
\end{array}
$$

and so on. We really omit many of the decimal points and zeros; these are useless if *we keep the places right;* notice that in order to do so two zeros are to be placed *after* the difference after subtraction, and one zero *after* the "*trial divisor.*"

The trial divisor is twice the amount previously found with the decimal point shifted as above.

The whole work in the above example would be written as follows:

$$
\begin{array}{r|l}
 & 1\dot{5}.0\dot{0}0\dot{0}0\dot{0}0\dot{0}\ \underline{|3.8729}, \text{ etc.} \\
 & 9 \\
\hline
2 \times 3 = 60 & 600 \\
\underline{8} & \\
68 & 544 \\
\hline
2 \times 38 = 760 & 5600 \\
\underline{7} & \\
767 & 5369 \\
\hline
2 \times 387 = 7740 & 23100 \\
\underline{2} & \\
7742 & 15484 \\
\hline
2 \times 3872 = 77440 & 761600 \\
\underline{9} & \\
77449 & 697041 \\
\hline
 & 64559, \text{ etc.}
\end{array}
$$

(The small dots placed over every other place beginning with the units' place in the original number are helpful in bringing down the figures, especially if the given number contains decimal digits.)

The result obtained is a decimal fraction, whose square is less than 15; but the square of the decimal just higher in the last place is greater than 15, at each stage:

$$(3.8)^2 < 15 < (3.9)^2,$$

$$(3.87)^2 < 15 < (3.88)^2,$$

$$(3.872)^2 < 15 < (3.873)^2.$$

We say that $\sqrt{15}$ is determined by this sequence of numbers, *i.e.* by the decimal fractions found as above.

(This is really the first definition of $\sqrt{15}$; the idea of drawing a smooth curve through the points we can locate in Fig. 31 contains the essentials of a strict definition. See Appendix, § 29.)

In carrying out any operation upon $\sqrt{15}$ we shall really operate upon these decimal fractions; the result is said to be the number defined by these decimal results. (See also Appendix, §§ 29–30.) The same statements apply, of course, to all inexact square roots as well as to $\sqrt{15}$.

Any inexact root, *i.e.* a root that cannot be expressed exactly in fractions or integers, is called a **surd**. Any expression containing a surd is called a **surd expression**. The quotient formed by dividing any integer by another integer is called a **rational fraction**. A number that is an integer or a rational fraction is often called a **rational number**. Any number that is neither an integer nor a rational fraction is called **irrational**. (See pp. 192, 284, and Appendix, § 29.) Thus, every *surd* is an *irrational* number.

EXERCISES I: CHAPTER VII

Find the square roots of the following numbers:

1. 441.	**4.** 1253.16.	**7.** .0625.
2. 324.	**5.** 24.1081.	**8.** .1369.
3. 55,225.	**6.** 27.6676.	**9.** .056644.

From the figure on p. 183 ($y = x^2$), read off the values of the following radicals:

10. $\sqrt{2}$. **11.** $\sqrt{6}$. **12.** $\sqrt{10}$. **13.** $\sqrt{18}$. **14.** $\sqrt{2.5}$.

Calculate the values of the following radical expressions to three places of decimals:

15. $\sqrt{2}$. **17.** $\sqrt{12}$. **19.** $\sqrt{19}$. **21.** $\sqrt{109}$.

16. $\sqrt{1.2}$. **18.** $\sqrt{33}$. **20.** $\sqrt{71}$. **22.** $\sqrt{2+\sqrt{3}}$.

23. $\sqrt{2-\sqrt{3}}$. **24.** $\sqrt{1-\sqrt{\sqrt{17}-4}}$.

98. Powers of Radicals. We are justified in saying that the square of $\sqrt{2}$ is 2, or, in general,* if a is positive,

$$(\sqrt{a})^2 = a.$$

If we do so, we can easily square such expressions as $3 + 2\sqrt{5}$.

For by p. 93, Chapter IV, we have

$$(x+y)^2 = x^2 + 2xy + y^2,$$

hence,
$$\begin{aligned}(3+2\sqrt{5})^2 &= (3)^2 + 2(3)(2\sqrt{5}) + (2\sqrt{5})^2\\ &= 9 + 12\sqrt{5} + 4(\sqrt{5})^2\\ &= 9 + 12\sqrt{5} + 20\\ &= 29 + 12\sqrt{5}.\end{aligned}$$

Check: $\sqrt{5} = 2.236$ (nearly).

Now $3 + 2\sqrt{5} = 7.472$ (nearly),

and $29 + 12\sqrt{5} = 55.832$ (nearly),

$(7.472)^2 = 55.832$ (nearly).

We shall need this process in the following chapter. Other operations of this kind will require some further explanation and are deferred until Chapter XI, p. 284.

* See, however, Chapter XI, p. 286, and Appendix, § 31.

EXERCISES II: CHAPTER VII

Square the radical expressions:

1. $2+\sqrt{3}$.
2. $2-\sqrt{3}$.
3. $1+3\sqrt{2}$.
4. $3\sqrt{5}-7$.
5. $\sqrt{8}-3$.
6. $2\sqrt{7}-5$.
7. $\frac{1}{2}\sqrt{3}-1$.
8. $x+\sqrt{y}$.
9. $x+\sqrt{a-x^2}$.
10. $1-\sqrt{x-1}$.
11. $\sqrt{17}-4$.
12. $2\sqrt{11}-7$.
13. $a\sqrt{b}-c$.
14. $mx+m\sqrt{y^2-x^2}$.
15. $kx-l\sqrt{y^2+1}$.
16. $kx+l\sqrt{y^2-\dfrac{k^2x^2}{l^2}}$.

99. Simple Operations. We shall also need to notice that $\sqrt{12}=\sqrt{4\cdot 3}=\sqrt{4}\sqrt{3}=2\sqrt{3}$ for example; or, in general, $\sqrt{a^2b}=\sqrt{a^2}\sqrt{b}=a\sqrt{b}$, if a and b are positive. This is justified by observing that

$$(2\sqrt{3})^2=4(\sqrt{3})^2=12, \quad \text{whence } 2\sqrt{3}=\sqrt{12};$$

or, in general, if a and b are positive

$$(a\sqrt{b})^2=a^2(\sqrt{b})^2=a^2b, \quad \text{whence } a\sqrt{b}=\sqrt{a^2b}.$$

Likewise,

$$\sqrt{\frac{3}{4}}=\frac{\sqrt{3}}{\sqrt{4}}=\frac{\sqrt{3}}{2};$$

or, in general,

$$\sqrt{\frac{b}{a^2}}=\frac{\sqrt{b}}{\sqrt{a^2}}=\frac{\sqrt{b}}{a}.$$

These operations are more fully studied in Chapter XI, p. 284.

EXERCISES III: CHAPTER VII

Remove from under the radical sign all square factors in the following, assuming that the letters used have positive values:

1. $\sqrt{18}$.
2. $\sqrt{50}$.
3. $\sqrt{a^3b^2}$.
4. $\sqrt{75}$.
5. $\sqrt{98}$.
6. $\sqrt{24}$.
7. $\sqrt{99}$.
8. $\sqrt{a^2xy^3}$.
9. $\sqrt{12pq^2}$.
10. $\sqrt{\frac{7}{16}}$.
11. $\sqrt{\frac{8}{9}}$.
12. $\sqrt{\frac{8}{27}}$.
13. $\sqrt{\dfrac{x^2y}{z^2t}}$.
14. $\sqrt{\dfrac{ab^2c^3}{d^2e^3f^4}}$.

15. $\sqrt{\frac{x}{y}}\ \left(=\sqrt{\frac{xy}{y^2}}\right)$.

16. $\sqrt{\frac{3}{10}}\ (=\sqrt{\frac{30}{100}})$.

17. $\sqrt{\frac{9}{10}}\ (=\sqrt{\frac{90}{100}})$.

18. $\sqrt{\frac{4}{5}}$.

Express the following quantities in a form entirely under the radical sign:

19. $5\sqrt{3}$.

20. $3\sqrt{5}$.

21. $6\sqrt{3}$.

22. $mn\sqrt{np}$.

23. $\frac{1}{6}\sqrt{3}$.

24. $\frac{5}{12}\sqrt{8}$.

25. $\frac{a}{b}\sqrt{bc}$.

26. $\frac{m}{n}\sqrt{\frac{n}{m}}$.

27. $\frac{x-y}{x+y}\sqrt{x^2-y^2}$.

100. Cube Roots. We may find cube roots by an analogous process. First let us draw a figure, where x means any number and y means the cube of that number, *i.e.* $y = x^3$. A table is

x	0	$+1$	-1	$+2$	-2	± 3	± 4	± 5	± 6	etc.	$\pm\frac{1}{2}$	$\pm\frac{1}{3}$	$\pm\frac{2}{3}$	$\pm\frac{3}{2}$	$\pm\frac{5}{2}$	$\pm\frac{7}{2}$	etc.
y	0	$+1$	-1	$+8$	-8	± 27	± 64				$\pm\frac{1}{8}$	$\pm\frac{1}{27}$	$\pm\frac{8}{27}$	$\pm\frac{27}{8}$			

[Let the student fill in the blank spaces and expand the table.]

From this table we may draw a figure that will appear as shown, the scale being taken for convenience:

1 vertically = 1 small space;

1 horizontally = 5 small spaces.

From this figure we can read off, approximately, the cube of any number, or also the cube root of any number, as for square roots above.

Thus, the cube of 1.8 is seen to be about $5\frac{3}{4}$ (really 5.832); the cube root of 11 is seen to be a little over 2.2. More accurate results can be found by drawing the figure on a larger scale.

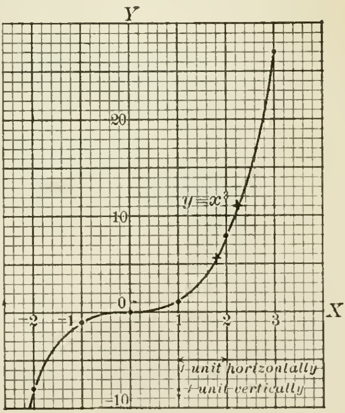


Fig. 33.

A method similar to that of § 97 can be devised for finding further decimal places. (See Appendix, § 20.) But it is really seldom used because there is a much quicker process. (See Chapter XIV on Logarithms, p. 339.)

101. Higher Roots. A graphical process like that above applies to finding any root of any number approximately. The figure shows the pictures for

$$(1) \quad y = x^4,$$

and

$$(2) \quad y = x^5,$$

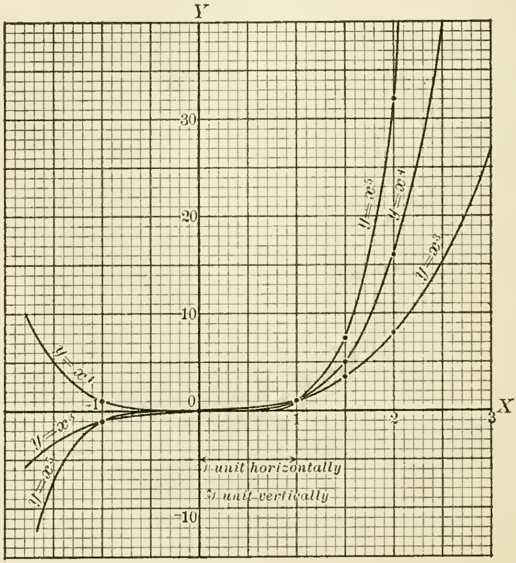


FIG. 34.

from which fourth and fifth roots may be found approximately. These roots may be found more accurately either

(1) By drawing a larger figure.

(2) By a method similar to that of § 97.

(3) By logarithms. (See Chapter XIV, p. 339.)

The last method is by far the easiest; further explanation of it will be found in the Chapter on Logarithms (Chapter XIV).

EXERCISES IV: CHAPTER VII

Draw accurately figures for $y=x^3$, $y=x^4$, $y=x^5$, as indicated on p. 190, and estimate as accurately as possible the values of the following radicals; check by raising the result to the proper power:

1. $\sqrt[3]{17}$. **3.** $\sqrt[5]{17}$. **5.** $\sqrt[5]{50}$. **7.** $\sqrt[4]{50}$. **9.** $\sqrt[4]{76}$.

2. $\sqrt[4]{17}$. **4.** $\sqrt[3]{25}$. **6.** $\sqrt[3]{34}$. **8.** $\sqrt[5]{60}$. **10.** $\sqrt[3]{100}$.

11. How many cube roots has a positive number among the numbers we already know? a negative number?

12. How many fourth roots has a positive number? a negative number?

13. How many fifth roots has a positive number? a negative number?

14. If $y=x^3$, what is x in terms of y? By noting corresponding values of x and y in Fig. 33 on p. 189, and plotting them reversed, draw the figure for $y=\sqrt[3]{x}$.

15. Draw the figure for $y=\sqrt[4]{x}$.

PART II. POWERS AND ROOTS OF MONOMIALS

102. Notation. Instead of writing $\sqrt{\ }$, $\sqrt[3]{\ }$, $\sqrt[4]{\ }$, ⋯, etc., for roots, we may write a small figure $\frac{1}{2}$ or $\frac{1}{3}$ or $\frac{1}{4}$ or $\frac{1}{5}$ ⋯, etc., at the upper right-hand corner, in the place usually occupied by an exponent.

$$\sqrt{4} = 4^{\frac{1}{2}} = 2, \quad -\sqrt{4} = -4^{\frac{1}{2}} = -2,$$

$$\sqrt[3]{8} = 8^{\frac{1}{3}} = 2, \quad \sqrt[3]{-8} = (-8)^{\frac{1}{3}} = -2.$$

So, in general, $\quad \sqrt[n]{\boldsymbol{a}} = \boldsymbol{a}^{\frac{1}{n}}$.

This notation is very convenient; it does not cause any confusion, for we have never written fractions in the exponent position before this; hence, there is nothing to be understood by $4^{\frac{1}{2}}$, for example, except $\sqrt{4}$.

Any expression that contains a root sign, either by use of the symbol $\sqrt{\ }$ or by means of a fraction in the exponent position, is called a **radical** expression.

103. Monomials. Monomials may be raised to powers, or their roots may be taken, according to the rules for multiplication.

Ex. 1. $(4\,ab^2)^3 = (4\,ab^2)(4\,ab^2)(4\,ab^2) = 64\,a^3b^6$. (See p. 73.)

Ex. 2. $16\,a^4b^2 = 4\,a^2b \cdot 4\,a^2b$; hence $\sqrt{16\,a^4b^2} = 4\,a^2b$.

Notice also $\qquad -\sqrt{16\,a^4b^2} = -4\,a^2b$.

In raising to powers or in taking roots, care must be taken with regard to signs. (See Rule of Signs, p. 71.)

Ex. 3. $\left(\dfrac{-2\,xy^3}{3\,a}\right)^2 = \left(\dfrac{-2\,xy^3}{3\,a}\right)\left(\dfrac{-2\,xy^3}{3\,a}\right) = \dfrac{+4\,x^2y^6}{9\,a^2}$.

Ex. 4. $\dfrac{-27\,a^3x^6}{8\,y^3} = \left(\dfrac{-3\,ax^2}{2\,y}\right)\left(\dfrac{-3\,ax^2}{2\,y}\right)\left(\dfrac{-3\,ax^2}{2\,y}\right)$;

hence, $\sqrt[3]{\dfrac{-27\,a^3x^6}{8\,y^3}} = \dfrac{-3\,ax^2}{2\,y}$; $\left(not\ \dfrac{+3\,ax^2}{2\,y}\right)$.

A **simple power** (p. 8) is one whose degree is a positive integer, such as 2, 3, etc. Thus, a^3 is a *simple* power.

EXERCISES V: CHAPTER VII

Perform the indicated involutions and evolutions:

1. $(a^2b)^3$.

2. $(-3\,xyz^2)^4$.

3. $(-2\,z^5t^3)^2$.

4. $(-mn^2p^5)^3$.

5. $(3\,x^ay^bz^c)^2$.

6. $\sqrt[3]{-64\,a^3b^6}$.

7. $\sqrt{x^2y^4z^{2n}}$.

8. $\sqrt[4]{81\,m^4n^{12}}$.

9. $\sqrt{\sqrt{625\,r^{8n}s^{12n}}}$.

10. $\sqrt[5]{-32\,x^{10}}$.

11. $\left(\dfrac{ax^2}{2\,bcy}\right)^3$.

12. $\left(\dfrac{-3\,rt}{ps^2v}\right)^2$.

13. $\left(\dfrac{-xy}{4\,z^3}\right)^5$.

14. $\sqrt{\dfrac{64\,m^4n^{2a}p^6}{9\,x^2y^{6b}z^{12c}}}$.

15. $\sqrt[3]{-\dfrac{27\,b^3c^{9r}}{a^{3s}x^{6t}}}$.

104. Formal Rule, Monomials. We may make a rule by the reasoning above. Thus, if m and n denote two positive integers,

I. $(x^n)^m = x^n \cdot x^n \cdots x^n (m \text{ times}) = x^{n\cdot m}$. (See rule, p. 72.)

To raise a letter with an exponent to a simple power, multiply the exponent by the degree of the power. In short, *in involution* **multiply** *exponents.*

It is well to contrast this strongly with the rule for multiplying powers, p. 73, which is

II. $x^n \times x^m = x^{n+m}$.

In multiplying, **add** *exponents.*

We also have

III. $(xy)^n = x \cdot y \cdot x \cdot y \cdot x \cdot y \cdots x \cdot y (n \text{ times})$

$= [x \cdot x \cdot x \cdots x(n \text{ times})] \times [y \cdot y \cdot y \cdots y(n \text{ times})] = x^n \times y^n.$

Any simple power of a product is equal to the product of the same powers of the separate factors.

III *a*. Similarly, $\frac{x^n}{y^n} = \left(\frac{x}{y}\right)^n$. This rule follows from III, also:

Any simple power of a quotient is equal to the quotient of the same powers of numerator and denominator.

These rules assist in reducing the work of *involution:*

To raise a monomial to a simple power, raise the coefficient to that power, multiply the exponent of each letter by the degree of the power, and make sure that the sign follows the Rule of Signs.

The Rule of Signs (p. 71) shows that

(1) an even number of negative factors gives + in the product;
(2) an odd number of negative factors gives − in the product;
(3) any number of positive factors gives + in the product.

This rule is simply the written expression of what we already did in § 103; *if this rule is forgotten, the examples can all be solved by* § 103.

105. Advantage of Fractional Notation. Practice with the fractional notation of § 102 shows its convenience.

Ex. 1. $\sqrt{16\, a^4b^2} = 4\, a^2b$, for $(4\, a^2b)(4\, a^2b) = 16\, a^4b^2$,

or, $(16\, a^4b^2)^{\frac{1}{2}} = 4\, a^2b$.

Note that this follows the Rules of § 104; for the work

$$(16\, a^4b^2)^{\frac{1}{2}} = (16)^{\frac{1}{2}}a^{4 \times \frac{1}{2}}b^{2 \times \frac{1}{2}} = 4\, a^2b$$

is correct, although we had no reason to expect it.

Ex. 2. Trying the same rules of § 103 for $\sqrt[3]{-27\,a^3x^6}$, we find:

$$(-27\,a^3x^6)^{\frac{1}{3}} = (-27)^{\frac{1}{3}}a^{3\times\frac{1}{3}}x^{6\times\frac{1}{3}} = -3\,ax^2,$$

which is correct, for $(-3\,ax^2)(-3\,ax^2)(-3\,ax^2) = -27\,a^3x^6$.

It is necessary for the student to check his work as above, to avoid errors, at least until he becomes expert.

Rule I of § 104 holds when the power is a fraction, that is, *for roots*, if the root can be found otherwise (*i.e.* for exact roots. See § 94, p. 182, and § 98, p. 187):

$$(a^{k\cdot n})^{\frac{1}{n}} = a^{k\cdot n\cdot\frac{1}{n}} = a^k,$$

for $$a^k\cdot a^k\cdots a^k(n \text{ times}) = a^{k\cdot n}.$$

Later (§ 136, p. 285), we shall see that the rules of § 104 always hold when we properly extend our notation.

Likewise, Rule III holds for exact roots:

$$(a^n)^{\frac{1}{n}} \times (b^n)^{\frac{1}{n}} = (a^nb^n)^{\frac{1}{n}},$$ for each side is equal to ab.

EXERCISES VI: CHAPTER VII

Perform the following involutions by the aid of Rule I:

1. $(-2\,a^2bc^3)^5$.
2. $(-xyz^5)^2$.
3. $(ab^2c^5)^3$.
4. $(-5\,mn^2p^3)^4$.
5. $(-ax^ry^{2s}z^{3t})^2$.
6. $(ab^2c^3)^m$.
7. $(x^{m-n}y^{n-p}z^{p-q})^a$.
8. $\left(\dfrac{-2\,x^2y^3z^5}{11\,a^mz^{p-q}}\right)^3$.
9. $\left(\dfrac{-5\,m^2n^3p}{3\,r^{a^2}s^{ab}}\right)^2$.
10. $\left(\dfrac{a^2b^{rs}c^{rs}}{x^{r-s}y^2}\right)^{rs}$.

Perform the following evolutions by the aid of Rule I; check by actual multiplication, when the index is numerical:

11. $\sqrt{64\,a^4b^6}$.
12. $\sqrt[3]{-64\,x^6y^9z^{12}}$.
13. $\sqrt[7]{2187\,p^7q^{21}}$.
14. $\sqrt[4]{625\,r^8s^{16}}$.

15. $\sqrt[3]{-27\,z^6}$.

16. $\sqrt[5]{\dfrac{-32\,a^{10}b^{15}}{243\,x^5y^{25}}}$.

17. $\sqrt[4]{\dfrac{256\,x^4y^{12}}{p^8z^4}}$.

18. $\sqrt[6]{a^{6m}c^{18}}$.

19. $\sqrt[m]{x^my^{2m}z^{mn}p^{m^2}q^{m^2-mn}}$.

20. $\sqrt[n]{\dfrac{a^{n^2}b^{np}c^{mn}}{x^{mn+n^2}y^{n-n^2r}}}$.

106. Longer Expressions. Binomials and longer expressions can be raised to powers by simply applying the rules for multiplication; or also by using the rules of Part III of Chapter IV, p. 91.

$(a \pm b)^2 = a^2 \pm 2\,ab + b^2$. (See p. 93.)

$(a \pm b)^3 = a^3 \pm 3\,a^2b + 3\,ab^2 \pm b^3$. (See p. 99.)

$(a \pm b)^4 = a^4 \pm 4\,a^3b + 6\,a^2b^2 \pm 4\,ab^3 + b^4$. (See p. 99.)

Ex. 1.

$$(4\,x^3y^2 - 3\,xy^3)^3 = (4\,x^3y^2)^3 - 3(4\,x^3y^2)^2(3\,xy^3) + 3(4\,x^3y^2)(3\,xy^3)^2 - (3\,xy^3)^3 = 64\,x^9y^6 - 144\,x^7y^7 + 108\,x^5y^8 - 27\,x^3y^9.$$

Test this by direct multiplication of three factors, each $4\,x^3y^2 - 3\,xy^3$.

Roots of longer expressions may be found by inspection, comparing with the formula written above.

Ex. 2. Find the square root of $4\,m^2 - 12\,mn + 9\,n^2$.

Comparing with $(a - b)^2$ we see $a = 2\,m$, $b = 3\,n$ will fit to give the given expression:

$$(2\,m - 3\,n)^2 = 4\,m^2 - 12\,mn + 9\,n^2;$$

hence, $$(4\,m^2 - 12\,mn + 9\,n^2)^{\frac{1}{2}} = (2\,m - 3\,n).$$

Another answer is $-(2\,m - 3\,n)$, by the general rule that the negative of one answer for a square root is also an answer; for,

$$[-(2\,m - 3\,n)]^2 = (-2\,m + 3\,n)^2 = 4\,m^2 - 12\,mn + 9\,n^2.$$

It is not certain which of these answers is the negative one until we know whether $2\,m - 3\,n$ is positive or negative. If $m = 5$ and $n = 1$, $2\,m - 3\,n = 7$; $(-2\,m + 3\,n) = -7$; the given expression is of course the square of this, *i.e.* 49.

Check: If $m = 5$ and $n = 1$,

$$4\,m^2 - 12\,mn + 9\,n^2 = 4(5)^2 - 12(5)(1) + 9(1)^2 = 100 - 60 + 9 = 49.$$

A numerical check of this kind is most valuable in avoiding errors.

Ex. 3. Find the square root of

$$4\,x^2 + 9\,y^2 + 16\,z^2 - 12\,xy + 16\,xz - 24\,yz.$$

Comparing with $a^2 + b^2 + c^2 + 2\,ab + 2\,ac + 2\,bc = (a + b + c)^2$, from p. 92, we see that $a = 2\,x$, $b = -3\,y$, $c = 4\,z$ will fit; for,

$$(2\,x - 3\,y + 4\,z)^2 = 4\,x^2 + 9\,y^2 + 16\,z^2 - 12\,xy + 16\,xz - 24\,yz.$$

Hence, $(4\,x^2 + 9\,y^2 + 16\,z^2 - 12\,xy + 16\,xz - 24\,yz)^{\frac{1}{2}} = 2\,x - 3\,y + 4\,z.$

Check: Put $\qquad x = 1,\ y = 2,\ z = 3.$

$$[4(1)^2+9(2)^2+16(3)^2-12(1)(2) + 16(1)(3) - 24(2)(3)]^{\frac{1}{2}} = 64^{\frac{1}{2}} = 8.$$

$$2\,x - 3\,y + 4\,z = 2(1) - 3(2) + 4(3) = 8.$$

Another answer is $-(2\,x - 3\,y + 4\,z) = -2\,x + 3\,y - 4\,z$. Check it.

EXERCISES VII: CHAPTER VII

Perform the indicated operations:

1. $(ax + by)^2$.

2. $(ab - xy)^2$.

3. $(2\,p - 7\,q)^3$.

4. $(3\,a - 2\,b)^5$.

5. $(m + 7\,n)^4$.

6. $(c - 2\,d + e)^2$.

7. $(bc + ca + ab)^2$.

8. $(2\,x - 3\,y + 4\,z)^2$.

9. $(ap - bq)^2 + (ab + pq)^2$.

10. $(x^4 - 8\,x^3 + 24\,x^2 - 32\,x + 16)^{\frac{1}{4}}$.

11. $(z^3 - 12\,z^2 + 48\,z - 64)^{\frac{1}{3}}$.

12. $(a^{2m} + 12\,a^m b^n + 36\,b^{2n})^{\frac{1}{2}}$.

13. $\left(r + \dfrac{2\,s}{3\,t}\right)^3$.

14. $\left(\dfrac{gt^2}{2} + vt\right)^2$.

15. $(4\,x^2 + y^2 + 9\,z^2 - 6\,yz + 4\,xy - 12\,zx)^{\frac{1}{2}}$.

107. Process for Square Roots. Roots of longer expressions may also be found by the following process, which is occasionally convenient. In simple examples, however, roots can be best found by inspection when there is any exact answer.

We know
$$(a+b)^2 = a^2 + 2\,ab + b^2$$
$$= a^2 + (2\,a + b)b.$$

This we write as follows:

$$\begin{array}{r|l}
 & a^2 + 2\,ab + b^2 \;\big|\; \underline{a+b} \\
 & \underline{a^2} \\
2\,a & \quad 2\,ab + b^2 \\
\underline{\quad + b} & \\
\underline{2\,a + b} & \quad \underline{2\,ab + b^2}
\end{array}$$

If there were more terms, we should proceed similarly; the *important step is to form the "trial divisor"* ($2\,a$, above), by multiplying the part already found by 2; to this we add the second term (b, above) and multiply the sum ($2\,a + b$, above) by this second term; this gives the total original amount as shown above.

Ex. 1. Find $(4\,m^2 - 12\,mn + 9\,n^2)^{\frac{1}{2}}$ (see Ex. 2, § 106, p. 196). The work may be written.

$$\begin{array}{r|l}
 & 4\,m^2 - 12\,mn + 9\,n^2 \;\big|\; \underline{2\,m - 3\,n} \\
 & \underline{4\,m^2} \\
2 \times 2\,m = 4\,m & -12\,mn + 9\,n^2 \\
\underline{-3\,n} & \\
\underline{4\,m - 3\,n} & \underline{-12\,mn + 9\,n^2}
\end{array}$$

Ex. 2. Find $(4\,x^2 + 9\,y^2 + 16\,z^2 - 12\,xy + 16\,xz - 24\,yz)^{\frac{1}{2}}$.

$$\begin{array}{r|l}
 & 4\,x^2 + 9\,y^2 + 16\,z^2 - 12\,xy + 16\,xz - 24\,yz \;\big|\; \underline{2\,x - 3\,y + 4\,z} \\
 & \underline{4\,x^2} \\
 & 9\,y^2 + 16\,z^2 - 12\,xy + 16\,xz - 24\,yz \\
2 \times 2\,x = 4\,x & \text{(arranged)} \\
\underline{-3\,y} & -12\,xy + 9\,y^2 + 16\,z^2 + 16\,xz - 24\,yz \\
\underline{4\,x - 3\,y} & \underline{-12\,xy + 9\,y^2} \\
2(2\,x - 3\,y) = 4\,x - 6\,y & 16\,z^2 + 16\,xz - 24\,yz \\
\underline{+4\,z} & \text{(arranged)} \\
4\,x - 6\,y + 4\,z & 16\,xz - 24\,yz + 16\,z^2 \\
 & \underline{16\,xz - 24\,yz + 16\,z^2}
\end{array}$$

In this example the terms cannot be arranged until we see which term will afford an exact quotient with the trial divisor. Practice will indicate the best arrangements.

Only *exact* square roots should be attempted by this process. The process is valuable also in discovering whether or not a given expression is a perfect square. Inexact roots are treated (for numbers only) in Part I, Chapter VII, § 97, pp. 184–186.

EXERCISES VIII: CHAPTER VII

Extract the square root of the following expressions:

1. $16\,p^2 + 25\,q^2 - 40\,pq.$

2. $9\,c^2 + 400\,d^2x^2 + 120\,cdx.$

3. $a^2 + 25\,b^2 + 9\,c^2 - 30\,bc + 6\,ca - 10\,ab.$

4. $x^4 + 2\,x^3y + 3\,x^2y^2 + 2\,xy^3 + y^4.$

5. $4\,x^4 - 12\,x^3 + 13\,x^2 - 6\,x + 1.$

6. $4 - 4\,a - 7\,a^2 + 4\,a^3 + 4\,a^4.$

7. $9\,x^4 - 12\,x^3 - 26\,x^2 + 20\,x + 25.$

8. $x^4 - 2\,x^3y - x^2y^2 + 2\,xy^3 + y^4.$

9. $a^6 - 2\,a^5 + 3\,a^4 - 4\,a^3 + 3\,a^2 - 2\,a + 1.$

10. $1 - a(6\,b + 4) + a^2(9\,b^2 + 12\,b + 4).$

11. $9\,a^6 - 12\,a^5b + 10\,a^4b^2 - 4\,a^3b^3 + a^2b^4.$

12. $49 - 42\,t + 37\,t^2 - 12\,t^3 + 4\,t^4.$

13. $4\,b^2c^2 - 16\,abc^2 + 16\,a^2c^2 - 12\,ab^2c + 24\,a^2bc + 9\,a^2b^2.$

14. $a^4 + 4\,a^3b + 6\,a^2b^2 + 4\,ab^3 + b^4.$

15. $r^6 - 6\,r^5 + 15\,r^4s^2 - 20\,r^3s^3 + 15\,r^2s^4 - 6\,rs^5 + s^6.$

REVIEW EXERCISES IX: CHAPTER VII

By use of the figures on preceding pages, estimate the following roots:

1. $\sqrt{60}$. **2.** $\sqrt[3]{60}$. **3.** $\sqrt[4]{60}$. **4.** $\sqrt[5]{60}$. **5.** $\sqrt{\sqrt{60}}$.

By rules for use of fractional exponents, so far as they have been given (§ 105, pp. 194–195), show that

6. $\sqrt{\sqrt{60}} = \sqrt[4]{60}$.

7. $\sqrt[3]{\sqrt{x}} = \sqrt[6]{x} = \sqrt{\sqrt[3]{x}}$.

8. $\sqrt[a]{\sqrt[b]{c}} = \sqrt[ab]{c} = \sqrt[b]{\sqrt[a]{c}}$.

9. $(\sqrt[k]{x})^k = x$.

Plot the following curves, making all necessary calculations by use of the figures (Figs. 32, 34, pp. 183, 190).

10. $y = x + x^2$.

11. $y = x - \sqrt{x}$.

12. $y = x + \sqrt[3]{x}$.

13. $y = \sqrt{x^3}$.

14. $y = \sqrt[3]{x^2}$.

15. $y = x^2 + \sqrt{x}$.

Perform the operations indicated:

16. $(5 - 2\sqrt{6})^2$.

17. $(5 + 2\sqrt{6})^2$.

18. $(5 + 2\sqrt{6})(5 - 2\sqrt{6})$.

19. $(5\sqrt{2} - 7)^2$.

20. $(5\sqrt{2} + 7)(5\sqrt{2} - 7)$.

21. $(3\sqrt{x} - 7y)^2$.

22. $(6x - y + 5z)^2$.

23. $(x + y + z)^3$.

24. $(27a^3 - 27a^2 + 9a - 1)^{\frac{1}{3}}$.

25. $(16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4)^{\frac{1}{4}}$.

Find the square root of each of the following expressions:

26. $9x^{10} + 6x^8 - 5x^6 - 2x^3 + 1$.

27. $x^4 - 6x^3y + 13x^2y^2 - 12xy^3 + 4y^4$.

28. $a^2 - 2ab - 3b^2 + \dfrac{4b^3}{a} + \dfrac{4b^4}{a^2}$.

29. $a^4 + b^2 + c^2 + d^4 + 2a^2b + 2a^2c + 2bd^2 + 2cd^2 + 2a^2d^2 + 2bc$.

30. $x^6 + 2x^5 + 3x^4 + 4x^3 + 3x^2 + 2x + 1$.

31. $a^6 - 4a^5 + 10a^4 - 20a^3 + 25a^2 - 24a + 16$.

SUMMARY OF CHAPTER VII: SIMPLE POWERS AND ROOTS, pp. 181–200

PART I. POWERS AND ROOTS OF NUMBERS. pp. 181–192.

Definitions: elementary definitions recalled; involution and evolution.

Number of Roots: odd roots, one answer; even roots of positives, two answers; even roots of negatives ("imaginary numbers"), no answers among numbers now known to student.

Notation: $\sqrt{\ }$ sign for even roots denotes the positive answer; the negative answer indicated by $-\sqrt{\ }$. § 94, pp. 181–182.

Figure for Square Roots: graph of $y = x^2$ drawn; square roots from figure. § 95, pp. 182–183.

Inexact Square Roots: approximate square roots of inexact squares found from figure. § 96, p. 184.

Formal Process for Square Roots: arithmetic process for closer approximation; formal process, using "trial divisor"; root actually defined.

Definitions: rational fraction — quotient of two integers; rational number — integer or rational fraction; irrational number — not rational; surd — irrational radical. Exercises I. § 97, pp. 184–187.

Squares of Radicals: $(\sqrt{a})^2 = a$; longer forms by previous rules. Exercises II. § 98, pp. 187–188.

Simple Operations on Quadratic Radicals: insertion and removal of simple factors; simple multiplications and divisions by integers. Exercises III. § 99, pp. 188–189.

Cube Roots: figure, $y = x^3$; approximate cube roots; other methods suggested. § 100, pp. 189–190.

Higher Roots: figures for x^4, x^5; other methods. Exercises IV. § 101, pp. 191–192.

PART II. SIMPLE POWERS AND ROOTS OF POLYNOMIALS. p. 192.

Fractional Notation for Roots: equivalence of $x^{\frac{1}{n}}$ to $\sqrt[n]{x}$; positive answer intended if n is even. § 102, p. 192.

Monomials: powers — direct extension of multiplication; roots — direct reversing of multiplication. Exercises V. § 103, pp. 192–193.

Formal Rules; Powers of Monomials: essentially, raise coefficient to power, multiply exponents. § 104, pp. 193–194.

Roots by Fractional Notation: correctness of rules of § 104 for exact roots in fractional notation. Exercises VI. § 105, pp. 194–196.

Longer Expressions: powers by previous rules; roots by inspection. Exercises VII. § 106, pp. 196–197.

Formal Process; Square Roots of Polynomials: illustrative problems; key is "trial divisor"; restricted usefulness. Exercises VIII. § 107, pp. 197–199.

Review Exercises for Chapter VII: Exercises VIII. p. 200.

CHAPTER VIII. QUADRATIC EQUATIONS

PART I. METHODS OF SOLUTION; CHARACTER OF THE ROOTS

108. Quadratic Equations. If an equation when cleared of fractions and radicals, and simplified, contains the square, but no higher power, of the unknown quantity, it is called a **quadratic** equation, or an equation of the second degree.

We have solved some such equations (see §§ 64–67, pp. 103–115, and § 82, p. 151); we shall now show how to solve any such equation.

Ex. 1. Given the equation $2x^2 - 9x + 4 = 0$.

We notice the factors $(2x - 1)$ and $(x - 4)$ on the left; hence we write

$$(2x - 1)(x - 4) = 0.$$

Whence,

$$2x - 1 = 0, \text{ or } x - 4 = 0,$$

and

$$x = \tfrac{1}{2}, \text{ or } x = 4.$$

If these factors were not noticed, or if the example were so difficult that the factors could not be seen by inspection, we could proceed as follows:

Trying various numbers for x, we should naturally try the numbers $x = 0$, $x = 1$, $x = 2$, etc., to see if we could find a correct answer by trial. Letting $x = 1$, for example, the left side is *not* zero; it is $2 - 9 + 4 = -3$.

Trying various numbers, as suggested, we should find a table like this:

x	0	1	2	3	4	5	6	7	etc.	-1	-2	-3	-4	etc.
l	4	-3	-6	-5	0	9	22		etc.	15				etc.

where l means the value of the left side of the given equation, that is,

$$l = 2x^2 - 9x + 4.$$

[Let the student fill in the blank spaces.]

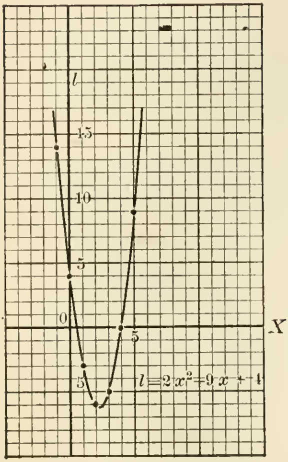


FIG. 35.

These values may be plotted in a figure, as on p. 183. The figure shows the value of the left side for any value of x; we wish to have the left side, l, equal to zero, in order to satisfy the given equation.

One value of x for which l is zero was discovered by trial on p. 203, namely, $x = 4$. It is clear from the figure that there is another value somewhere between 0 and 1, for *the curve crosses the horizontal line*, i.e. *the left side is zero*, somewhere between 0 and 1, about .5.

These answers are correct, as we found by a different method.

This process will not always give an accurate answer, of course, because the figure is not entirely accurate.

EXERCISES I: CHAPTER VIII

[The first of these exercises are easy examples by the factoring method; the student should solve them by factoring as on pp. 107 and 151. Then he should draw the figure as illustrated above and compare the results.]

1. $x^2 - 7x + 6 = 0.$

2. $r^2 - r - 12 = 0.$

3. $t^2 = 3t + 10.$

4. $z^2 = 9z - 20.$

5. $2t^2 - 11t + 12 = 0.$

6. $3t^2 - 7t - 6 = 0.$

7. $13p = 2p^2 + 15.$

8. $2m^2 + m = 21.$

9. $10t^2 + t - 3 = 0.$

10. $3v^2 = 34v - 40.$

[The second half of these exercises includes examples that the student will find too difficult by the factoring method. In these he should only draw the figure and estimate the answers.]

11. $n^2 - 4n + 2 = 0.$

12. $m^2 - 6m + 6 = 0.$

13. $t^2 - 5t + 3 = 0.$

14. $x^2 - 7x - 4 = 0.$

15. $2x^2 - 3x - 6 = 0.$

16. $3z^2 - 7z + 3 = 0.$

17. $2r^2 + 3r - 1 = 0.$

18. $x^2 - x - 1 = 0.$

109. General Solution. If an absolutely precise answer is needed, it can be found by the process illustrated below, no matter how difficult the example may be by factoring.

Ex. 1. $x^2 - 4x - 5 = 0.$

Transpose 5 and write, $x^2 - 4x = 5.$

Add 4 to each side: $x^2 - 4x + 4 = 9.$

The purpose of this is to make the expression on the left side a perfect square. Evidently it is $(x - 2)^2$, and we have:

$$(x-2)^2 = 9.$$

Whence, $x - 2 = \pm 3,$

and $x = 2 \pm 3 = 5$ or $-1.$

Check: For $x = 5$,

$$5^2 - 4 \cdot 5 - 5 = 0 \text{ (correct).}$$

For $x = -1$,

$$(-1)^2 - 4(-1) - 5 = 1 + 4 - 5 = 0 \text{ (correct).}$$

Problems should always be carefully checked. An *answer* is a value of x (or whatever letter is used) that satisfies the original equation when x is replaced by it. Such an answer is often called a **solution**, or a **root** of the equation.

Ex. 2. $2x^2 - 9x + 4 = 0.$ (See Ex. 1, p. 203.)

Divide by 2 on both sides, and transpose 4:

$$x^2 - \tfrac{9}{2}x = -2.$$

Desiring to make a perfect square on the left side we *add to each side the square of* $\frac{1}{2}$ *of the coefficient of* x, that is $(\frac{1}{2} \times -\frac{9}{2})^2 = \frac{81}{16}$,

$$x^2 - \tfrac{9}{2}x + \tfrac{81}{16} = -2 + \tfrac{81}{16},$$

or, $(x - \frac{9}{4})^2 = \frac{49}{16}.$

Whence, $x - \frac{9}{4} = \pm \frac{7}{4},$

or, $x = \frac{9}{4} \pm \frac{7}{4} = 4$ or $\frac{1}{2}.$

Check: For $x = 4$; $2 \cdot 4^2 - 9 \cdot 4 + 4 = 0$ (correct).

For $x = \frac{1}{2}$; $2 \cdot (\frac{1}{2})^2 - 9 \cdot (\frac{1}{2}) + 4 = 0$ (correct).

These are the same answers as those found above for this problem.

110. Completing a Square. Remembering the rule

$$(x + a)^2 = x^2 + 2\,ax + a^2,$$

we notice that the last term (a^2) in a perfect square is the square of $\frac{1}{2}$ the coefficient of x, if the first term is x^2. The preceding work depends on this, and we may definitely state the rule:

To solve a quadratic equation:

(1) *Notice what letter denotes the unknown quantity; we shall here call it x.*

(2) *Transpose the terms in x^2 and in x to the left side of the equation, the other term or terms to the right.*

(3) *Divide both sides by the coefficient of x^2.*

(4) *Add to each side the square of half the coefficient of x, so that the left side is a perfect square.*

(5) *Extract the square root of both sides, taking care to put both signs $\pm$ on the right.*

(6) *Solve the resulting equations by transposing to one side all but the term in x.*

(7) *Check each answer by substitution in the original equation.*

In step (5) the sign $\pm$ might be put on *both* sides, since a square root is extracted on *each* side. But the effect is the same as that given in (5), since any equation $\pm A = \pm B$

with $\pm$ on *both* sides is really the same as the equation

$$A = \pm B$$

with $\pm$ on one side only.

[The student should now carefully examine the solutions of the problems solved in § 109 and notice that this rule is followed there.]

The solution may not come out in rational form, as illustrated by the following example:

Ex. 1. $x^2 - 4x + 1 = 0.$

Transpose $+1$ and add $(\frac{1}{2} \cdot 4)^2$ or 4 to each side:

$$x^2 - 4x + 4 = 3.$$

Take the square root of each side:

$$x - 2 = \pm\sqrt{3},$$

or, $$x = 2 \pm \sqrt{3} = 2 + \sqrt{3}, \text{ or, } 2 - \sqrt{3}.$$

Check: For $2 + \sqrt{3}$:

$$(2+\sqrt{3})^2 - 4(2+\sqrt{3}) + 1 = 4 + 4\sqrt{3} + (\sqrt{3})^2 - 8 - 4\sqrt{3} + 1 = 0.$$

For $2 - \sqrt{3}$: (Correct.)

$(2-\sqrt{3})^2 - 4(2-\sqrt{3}) + 1 = 0.$ (Correct; student finish the work.)

The figure is drawn as in § 108.

x	0	1	2	3	4	5	6	7	8	etc.	-1	-2	-3	-4	etc.
l	1	-2	-3	-2	1	6					6				

From the figure the answers are seen to be

$$x = 0.3 \text{ (about)},$$

and $$x = 3.7 \text{ (about)}.$$

This compares well with the results above, for $\sqrt{3} = 1.73^+$; hence the answers found above are

$$x = 2 - 1.73^+ \text{ and } x = 2 + 1.73^+,$$

or, $$x = .27^+ \quad \text{and } x = 3.73^+.$$

It is seen that the figure shows approximately these square root values.

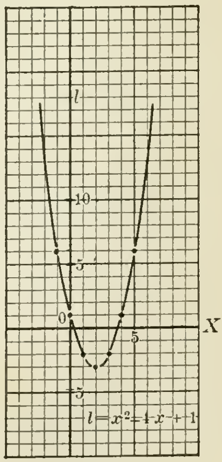


FIG. 36.

EXERCISES II: CHAPTER VIII

[In solving these exercises, the student should draw a figure for each one as in § 108. The answers found from the figure will serve as a check on the correctness of the work of solution.]

1. $x^2 - 15x + 44 = 0.$

2. $p^2 - 15p + 54 = 0.$

3. $a^2 - 15a = 100.$

4. $x^2 - 15x = 76.$

5. $d^2 - 15d = 154.$

6. $2r^2 - r = 36.$

7. $10n^2 - 31n + 24 = 0$.	**17.** $m^2 + 6m = 3$.
8. $3a^2 - 11a + 10 = 0$.	**18.** $p^2 - 10p = 5$.
9. $4x^2 + 4x = 15$.	**19.** $x^2 + x = 7$.
10. $x^2 - 7x + 3 = 0$.	**20.** $x^2 + 3x = 10$.
11. $x^2 - 5x + 3 = 0$.	**21.** $2x^2 + 5x = 4$.
12. $n^2 - 3n - 3 = 0$.	**22.** $3p^2 + 4p = 6$.
13. $x^2 - x - 3 = 0$.	**23.** $a^2x^2 + 2abx = a^2 - b^2$. (Do not plot figure.)
14. $q^2 + q - 3 = 0$.	**24.** $m^2 + 11m = 3$.
15. $r^2 + 3r - 3 = 0$.	**25.** $6x^2 + 3x = 10$.
16. $a^2 + 2a = 14$.	

Also solve the exercises in List I, pp. 204–205, by the method of § 110 and compare with the answers found before.

111. Second Method. Another method slightly different from the preceding will now be illustrated by the example used in § 109 (Ex. 2).

Ex. 1. $2x^2 - 9x + 4 = 0$.

Multiply each side by 8, and transpose 32:

$$16x^2 - 72x = -32,$$

or,

$$(4x)^2 - 18(4x) = -32.$$

Add the square of $\frac{1}{2}$ the coefficient of $4x$ in the middle term, that is, the square of $\frac{1}{2}$ of 18, or 9^2:

$$(4x)^2 - 18(4x) + (9)^2 = -32 + (9)^2,$$

or,

$$(4x - 9)^2 = 49;$$

therefore,

$$4x = 9 \pm 7 = 16 \text{ or } 2,$$

whence,

$$x = 4 \text{ or } \tfrac{1}{2}.$$

These are the same answers as were found before.

This method is practically the same as the previous one, except that $4x$ is used instead of x in performing the operations.

This method is sometimes more convenient, since long fractions may be avoided. It will be enough to *start by multiplying both sides by four times the coefficient of* x^2, so that the term in x^2 is an easy perfect square.

This method is often called the Hindu method, owing to its origin in India. In many cases the work may be shortened by multiplying by a smaller number than four times the first coefficient. Thus, the solution of the preceding example may be shortened somewhat by multiplying by 2 instead of by 8 and then solving in terms of $(2x)$ in place of $(4x)$. Let the student do this.

It is best to use the method previously given as the standard method, and to use this new one only rarely.

EXERCISES III: CHAPTER VIII

Solve the following examples both by the method of § 110 and by the method of § 111, comparing the two solutions for ease and accuracy. Abbreviate the latter method when possible. Always draw a figure as a check.

1. $2x^2 - 11x + 12 = 0$.

2. $2p^2 - 3p - 5 = 0$.

3. $2z^2 - 17z - 9 = 0$.

4. $3t^2 - 11t + 6 = 0$.

5. $3t^2 - 11t - 4 = 0$.

6. $3q^2 + q - 10 = 0$.

7. $4n^2 - 16n + 15 = 0$.

8. $4n^2 - 23n + 15 = 0$.

9. $6L^2 - L - 15 = 0$.

10. $5x^2 - x - 18 = 0$.

11. $3t^2 - 8t - 7 = 0$.

12. $4z^2 - 3z - 3 = 0$.

13. $2r^2 - r - 4 = 0$.

14. $3r^2 - 2r - 3 = 0$.

15. $5m^2 - 3m - 4 = 0$.

16. $9x^2 - 10x - 3 = 0$.

17. $7x^2 - 12x + 3 = 0$.

18. $3z^2 - 7z + 3 = 0$.

112. Equal Roots. Instead of having two different answers, as in most of the examples above, *a quadratic equation may have only one answer.*

Ex. 1. $x^2 - 4x + 4 = 0$.

The left side is *already* a perfect square; taking the square root of each side:

$$x - 2 = 0, \text{ or } x = 2,$$

since $\sqrt{0}$ is 0.

This equation has *only one solution.* It is clear that this will be true whenever *all the terms form a perfect square, after they have been transposed to one side.*

It is sometimes said, in this case, that the equation has *two equal roots*, in order to keep up the notion of two roots.

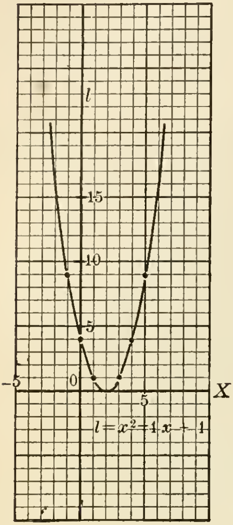


FIG. 37.

The figure in this case has only one point on the horizontal line, for there is only one value of x which makes the left side $= 0$.

The corresponding values of x and of $l = x^2 - 4x + 4$ are:

x	0	1	2	3	4	5	etc.	-1	etc.
l	4	1	0	1	4		etc.	9	etc.

Plotting these values in a figure (Fig. 37), as in § 108, we find that only one point, the point for which $x = 2$, is on the horizontal line. Hence there is but one answer, $x = 2$, as found above.

113. Imaginary Roots. It may also happen that a quadratic equation has no solution.

Ex. 1. $x^2 + 2 = 0$.

Drawing the figure, as before, we find:

x	0	1	2	etc.	-1	-2	-3	etc.
l	2	3	6		3	6	11	etc.

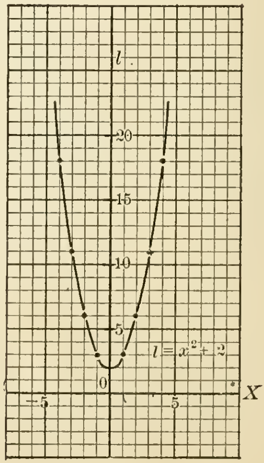


FIG. 38.

From the figure, and from the work done in making the table, it is clear that *there is no value of x* among the numbers familiar to us for which the left side is zero; for the smallest value of l is 2 (for $x = 0$), and the value of l rises steadily from this lowest value on both sides.

Ex. 2. $x^2+2x+5=0$.

Drawing as before, we find:

x	0	1	2	3	4	etc.	-1	-2	etc.
l	5	8	13	20			4	5	etc.

Here, again, the value of l is never zero; it is smallest for $x=-1$ and rises steadily on each side of $x=-1$.

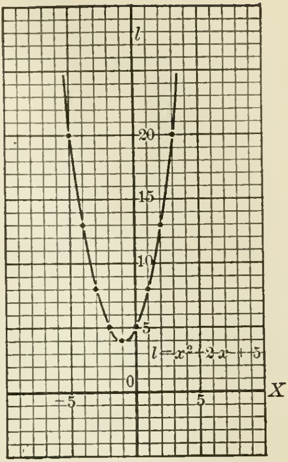


Fig. 39.

These examples show that it is possible for a quadratic equation to have no solution among numbers we know. Care must be taken to draw the figure very accurately, for a slight error might make it appear that the figure for example 1, § 112, did *not* have a point on the horizontal line.

Referring to Ex. 1, the numerical work may be done as follows:

Ex. 1. $x^2+2=0$.

Transpose 2: $\qquad x^2=-2$ or $x=\sqrt{-2}$.

But there is no ordinary number whose square is equal to -2, for the squares of the numbers we know are all positive. Hence, there is no solution.

Ex. 2. $x^2+2x+5=0$.

Proceeding as if we were solving by the usual method, we transpose 5 and add the square of $\frac{1}{2}\cdot 2$ to each side:

$$x^2+2x+1=-4,$$

or, $$(x+1)^2=-4, \text{ or } x=-1\pm\sqrt{-4}.$$

But this answer is meaningless, for we know no number whose square is -4. In such an example the expressed answers (in this example $-1\pm\sqrt{-4}$) are meaningless to the student at present; they are often called *imaginary* solutions. Later on (see Appendix, § 31), a certain useful meaning will be given to them.

There follow several examples, some of which have just one solution, and some, no solution.

EXERCISES IV: CHAPTER VIII

[Draw the figure in each case; also make an attempt at solution, as above.]

1. $p^2 - 5p + 8 = 0$.
2. $x^2 - 4x + 5 = 0$.
3. $x^2 - 3x + 4 = 0$.
4. $t^2 - 6t + 9 = 0$.
5. $z^2 + 7z + 13 = 0$.
6. $m^2 - 9m + 21 = 0$.
7. $x^2 - 8x + 16 = 0$.
8. $v^2 - 3v + 3 = 0$.
9. $z^2 - 4z + 8 = 0$.
10. $r^2 - 8r + 20 = 0$.
11. $2L^2 - 7L + 7 = 0$.
12. $3m^2 - 10m + 9 = 0$.
13. $4n^2 - 12n + 9 = 0$.
14. $5p^2 - 6p + 3 = 0$.
15. $2x^2 - 4x + 3 = 0$.
16. $9s^2 - 30s + 25 = 0$.
17. $t^2 - 12t + 36 = 0$.
18. $x^2 - 7x + 13 = 0$.
19. $2y^2 - 5y + 4 = 0$.
20. $3m^2 - 4m + 3 = 0$.

114. General Solution by Formula. We shall now express the solution of any quadratic equation. The student has seen above that we could always do so, though the expressed solution may not have a meaning.

Any quadratic equation is of the form:

$$ax^2 + bx + c = 0,$$

where a, b, c are fixed, known numbers in any one example, and a is not zero. Divide both sides by a, as on p. 206.

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$$

Transpose $\frac{c}{a}$ and add $\left(\frac{b}{2a}\right)^2$ to both sides:

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}.$$

The left side is found to be the square of $\left(x + \frac{b}{2a}\right)$:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

Taking the square root of each side, we find

$$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}=\pm\frac{\sqrt{b^2-4ac}}{2a}.$$ (See § 99, p. 188.)

Transposing $\frac{b}{2a}$, we find

$$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a},$$

or,

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

Check: These answers are correct, for we may reverse each step, starting with the answers and going backwards through the work to the original equation. Let the teacher guide the students in doing this, accounting for each step as it is taken.

The answers may also be checked by direct substitution in the given equation, $ax^2+bx+c=0$. This work is not written here because it is long. The teacher should guide the students in actually doing it.

Ex. 1. $x^2-4x-5=0$. (See Example 1, § 109.)

Comparing with $ax^2+bx+c=0$, we find

$$a=1,\ b=-4,\ c=-5.$$

Putting these numbers in the general result just found, we get

$$x=\frac{-(-4)\pm\sqrt{(-4)^2-4(1)(-5)}}{2(1)}$$

$$=\frac{+4\pm\sqrt{36}}{2}=\frac{4\pm 6}{2}=-1 \text{ or } +5,$$

which are the answers found before.

Ex. 2. $2x^2-9x+4=0$. (See example 2, § 109.)

$$a=2,\ b=-9,\ c=4.\quad \therefore b^2-4ac=(-9)^2-4(2)(4)=49,$$

$$x=\frac{-(-9)\pm\sqrt{49}}{2\cdot 2}=\frac{9\pm 7}{4}=4 \text{ or } \tfrac{1}{2}.$$

It is often best, as here, first to calculate the value of b^2-4ac, the quantity under the radical sign. One reason for this is seen in the next article.

The other examples solved in the text above are given in the following table, solved by the preceding formula.

No	Example	a	b	c	b^2-4ac	x (not reduced)	x (reduced)
3	$x^2-4x+1=0$	1	-4	1	12	$(4\pm\sqrt{12})\div 2$	$2\pm\sqrt{3}$
4	$x^2-4x+4=0$	1	-4	4	0	$(4\pm\sqrt{0})\div 2$	2 (no other)
5	$x^2+2=0$	1	0	2	-8	$(0\pm\sqrt{-8})\div 2$	meaningless
6	$x^2+2x+5=0$	1	2	5	-16	$(2\pm\sqrt{-16})\div 2$	meaningless

115. Discriminant. After a little practice it will be noticed that the value of b^2-4ac shows the nature of the result.

A. 1. If $b^2-4ac>0$, *there are two answers* ("*unequal real roots*").

(In a figure, drawn as in § 108, the curve cuts the main horizontal in two points.)

A. 2. If $b^2-4ac=0$, *there is only one answer* ("*equal roots*").

(In a figure, drawn as in § 112, the curve just touches the main horizontal line.)

A. 3. If $b^2-4ac<0$ *there are no answers* ("*imaginary roots*").

(In a figure, drawn as in § 113, the curve does not cut the main horizontal.)

For b^2-4ac is the quantity under the radical sign; if it is negative we have the square root of a negative quantity, which is meaningless to the student at present.

B. If b^2-4ac is an exact perfect square, the answers are rational; otherwise, the answers are irrational, provided a, b, c are rational.

On account of these facts the quantity $b^2 - 4ac$ is often called the **discriminant**; the knowledge of its value enables us to *discriminate* among the cases mentioned.

Notice that if these rules are forgotten, all this information can be found in any one example by attempting to solve as in §§ 109–113, by completing the square.

EXERCISES V: CHAPTER VIII

Draw the figure, solve by completing the square, and also solve by the formula. Compare the three answers.

1. $x^2 - 10x + 16 = 0.$

2. $3x^2 + 5x - 12 = 0.$

3. $2z^2 - 9z + 4 = 0.$

4. $3y^2 - 8y + 5 = 0.$

5. $2t^2 + t - 15 = 0.$

6. $6u^2 - 13u + 6 = 0.$

7. $3v^2 - 22v + 7 = 0.$

8. $4m^2 - 20m + 25 = 0.$

9. $6r^2 + r - 2 = 0.$

10. $5z^2 - 3z - 36 = 0.$

11. $3n^2 - 14n - 5 = 0.$

12. $5p^2 - 17p + 6 = 0.$

13. $7x^2 - 10x + 3 = 0.$

14. $9L^2 - 30L + 25 = 0.$

15. $8k^2 - 2k - 15 = 0.$

16. $8x^2 - x - 30 = 0.$

Find by calculating $b^2 - 4ac$ whether the roots are (A) real and unequal, equal, imaginary; (B) rational or irrational, without actually finding the roots.

17. $5r^2 - 7r - 6 = 0.$

18. $4s^2 - 28s + 49 = 0.$

19. $2m^2 + m - 21 = 0.$

20. $3p^2 + 4p - 6 = 0.$

21. $2x^2 - 11x + 12 = 0.$

22. $3p^2 - 7p + 5 = 0.$

23. $3r^2 - 7r + 4 = 0.$

24. $s^2 + 9s + 10 = 0.$

25. $4y^2 - 11y + 9 = 0.$

26. $16m^2 - 8m + 1 = 0.$

27. $s^2 - 11s + 32 = 0.$

28. $t^2 + 8t + 15 = 0.$

29. $4v^2 - 12v + 9 = 0.$

30. $6x^2 + 13x + 8 = 0.$

31. $3y^2 - y - 1 = 0.$

32. $4r^2 - 7r + 5 = 0.$

33. $3L^2 - L + 1 = 0.$

34. $x^2 - (m + 1)x + m = 0.$

35. $as^2 + (a + b)s + b = 0.$

36. $4a^2x^2 + 4ax + 1 = 0.$

PART II. PRACTICAL APPLICATIONS; PROBLEMS

116. Practical examples have been given that lead to quadratic equations (see pp. 103–112). The following may now be solved:

EXERCISES VI: CHAPTER VIII

After forming the equations, solve, and draw the graph:

1. The sum of the squares of two consecutive integers is 61. What are the numbers?

2. A number is less by 6 than the third of its square. What is the number?

3. Find two consecutive integers whose product is 462.

4. What numbers differing by 7 have the product 60?

5. Find a number of two digits, whose tens' digit exceeds its units' digit by one, if the sum of the digits times the original number is 63.

6. Find a number that exceeds twice its square by $\frac{2}{25}$.

7. Find three consecutive integers such that the sum of the squares of the first two is the square of the third.

8. Find a number whose square multipled by 3 exceeds 20 times the number by 7.

9. Find two numbers whose sum is 12, and the sum of whose squares is 74.

10. Find three consecutive odd integers such that the first two taken in order as the digits of a number express the product of the last two. Solve the same problem in even integers.

11. A stream flows at the rate of 5 miles an hour; a crew rows 6 miles with the stream and the same distance back in $3\frac{1}{5}$ hours. What is the rate of the boat in still water?

12. A crew rows upstream against a current of 3 miles an hour, for a distance of 8 miles; then back again. If the trip takes 5 hours, what rate could the crew make in still water?

13. A crew rows upstream $4\frac{1}{4}$ miles against a current of 4 miles an hour; it drifts back 2 miles, and rows at the original rate to the starting point. What is the rate of rowing, if the trip is made in 5 hours?

14. A crew able to make 3 miles an hour in still water rows 8 miles upstream and back in a total time of 6 hours. What is the rate of the current?

15. A crew able to make 5 miles an hour in still water rows 6 miles upstream and drifts back in a total time of 5 hours. How fast is the current flowing?

16. An open box, to be made from a square piece of cardboard, as in example 1, p. 103, by cutting out a four-inch square from each corner and turning up the sides, is to contain 256 cubic inches. How large a square must be used?

17. How large a square should be cut from each corner of a piece of tin 18 in. square, to form an open box whose total surface area is 260 sq. in., by turning up the sides?

18. A piece of cardboard 5 in. longer than wide is used to make a box of capacity 108 cu. in. by the method of Exs. 16, 17. How large a card must be used if 3-inch squares are cut from the corners?

19. From a card three times as long as wide a box containing 56 cu. in. is made as above by cutting half-inch squares from the corners. Find the size of the card.

20. From what size of card, two thirds as wide as it is long, can a box containing 125 cubic inches be made by cutting out squares of side $2\frac{1}{2}$ inches from the corners?

21. If the error, in inches, made in measuring the side of a square 10 ft. long is denoted by e, and the resulting error, in square inches, in the computed area by E, $E = e^2 + 240\,e$. (See example 2, p. 108.) Find E if $e = 1$; 3; $\frac{1}{2}$; 0; -1; etc. Plot the graph. Find e if $E = 484$; if $E = 1000$; if $E = -711$. Solve for e in terms of E, in general.

22. In Ex. 21, find E in terms of e if the side of the square is 25 ft. long. Plot a figure. Find e if $E = 2500$. Solve for e in terms of E, in general.

23. In Ex. 22, how nearly must the side be measured to make the computed area correct to within 2 sq. ft.? If a foot rule is used, how accurately must the ruler be placed each time? Is this practicable? See Exs. 51–53, p. 115.

24. If, in Ex. 21, the square is replaced by a circle whose radius is 10 ft., answer the same questions asked in Ex. 21, assuming that the *radius* is measured. See Ex. 18, p. 112.

Boyle's Law states that under constant temperature the product of the volume, v, and pressure, p, of a quantity of gas remains constant. This is true whatever units of measurement are used; we use "centimeters of mercury" for pressure and liters for volume. The product "pv" in those units will be called K.

25. A mass of hydrogen for which $K = 85{,}120$ is confined in a collapsible reservoir of unknown capacity. An increase of the pressure by 4 centimeters causes the reservoir to collapse partially, decreasing the volume by 56 liters. What was the original pressure? The original volume?

Solution. Let p be the original pressure. Then $\dfrac{85120}{p}$ is the original volume. The new pressure is $p + 4$, the new volume $\dfrac{85120}{p} - 56$. Hence,

$$(p+4)\left(\frac{85120}{p} - 56\right) = 85120,$$

$$p^2 + 4p - 6080 = 0,$$

or, on solving, $$p = -80 \text{ or } 76.$$

The solution of the problem is $p = 76$ centimeters; also, we see easily that $v = 1120\ l$.

26. For a certain mass of gas in a collapsible reservoir, $K = 277{,}500$. A decrease of 1 cm. in pressure causes an increase of 50 l. in volume. Find the original volume and pressure.

Solve the following similar problems:

27. $K = 162,060$. Increase in p, 1 cm. Decrease in v, 30 l.

28. $K = 275,625$. Increase in p, 1.5 cm. Decrease in v, 75 l.

29. $K = 112,480$. Decrease in p, 2 cm. Increase in v, 40 l.

30. Two tanks of equal capacity are emptied by unequal pipes; it is observed that the tank having the larger pipe is empty two hours sooner than the other. Both pipes attached to a single tank fill it in $2\frac{2}{5}$ hours. How long do the pipes separately require to fill or empty the tanks?

31. Two equal tanks are filled by pipes, one requiring 3 hours longer than the other. Both pipes together fill one tank in 2 hours. How long does each pipe require to fill one tank?

32. Two pipes are used, the larger to fill a tank, the smaller to empty it. When they are both open, the tank is filled in 15 hours. The pipes are then detached and used separately to empty two tanks each equal to the first. It is observed that this work is done 4 hours sooner by the large than by the small pipe. Find the number of hours required by each pipe.

33. A tank is filled by a certain pipe in an hour less than is required for a larger pipe to fill a tank of twice the capacity of the first. Both pipes together fill the large tank in 3 hr. 45 min. In what time could each pipe fill the small tank?

34. Two tanks whose capacities are as 2 to 3 are emptied by pipes, the larger tank requiring an hour longer than the other. Both pipes together fill the larger tank in $1\frac{1}{2}$ hr. How long would each separately require to fill the small tank?

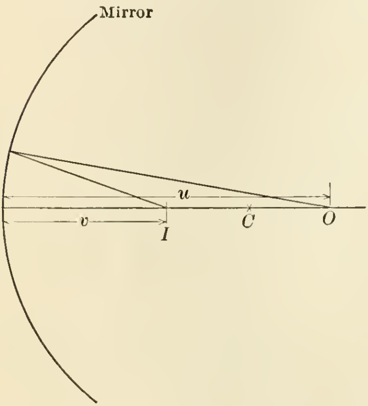
35. Two men are on streets at right angles to each other, distant 7 and 8 feet from the crossing. If they approach the corner at the same rate, how far must each walk so that their distance apart shall be 5 feet? Comment on the two answers.

36. Two men standing 1 foot and 8 feet from the crossing of two streets at right angles to each other walk away from the crossing at the same rate. When will they be 13 feet apart? Interpret the two answers.

37. Two men stand on streets meeting at right angles, in positions 3 and 5 feet from the crossing. How far must each walk toward the crossing at the same rate in order that they may be 10 feet apart? Interpret the two answers. Here neither really satisfies the implied conditions of the problem.

38. Two electric cars are on tracks meeting at right angles. One starts from the crossing at the rate of 20 feet a second, the other starts 100 feet away from the crossing at the rate of 10 feet a second. When will they be 200 feet apart?

A concave mirror is formed from a part of a spherical shell of radius r. If the distance of an object (O) from the mirror is denoted by u, and the distance of the image (I) of the object from the mirror is denoted by v, it is known that approximately



$$\frac{1}{u}+\frac{1}{v}=\frac{2}{r}.$$

(The proof of this formula need not be attempted; being an approximation, its derivation is not a direct proof.) Note that image and object are situated on opposite sides of the center (C) of the mirror, also that placing the object where the image was, throws the image where the object was.

39. How far is an object from a concave mirror of radius 20 cm. if its reflection is 15 cm. farther from the mirror?

40. Where are the object and its reflection in the mirror of Ex. 39, if their distance apart is 48 cm.?

41. What radius must be chosen for a concave mirror in order that an object $7\frac{1}{2}$ cm. from the mirror may be reflected 5 cm. farther from the mirror than the center?

42. An object is placed beyond the center of a concave mirror so that the distance from the object to the center of the mirror shall be twice the distance from the center to the reflection. Where is the object placed? (State the result in terms of the radius of the mirror.) Comment on the two results.

43. I desire to place an object before a concave mirror so that the center of the mirror may lie halfway between the object and its image. Can I do so?

44. A cylindrical box 6 inches high, open at the top, has a surface of $64\,\pi$ sq. in. What radius must be chosen?

45. If the box of Ex. 44 is 6 in. high and if the radius of its base is 4 in., show that an error of e (in in.) in measuring the radius causes an error E (in sq. in.) in the total surface, such that $E = \pi\,(20\,e + e^2)$. Find e if $E = 21\,\pi$; if $E = 5\,\pi$.

46. About how much error e would be caused if we took $\pi = 3$ in place of $\pi = 3\frac{1}{7}$ in the value $64\,\pi$ mentioned in Ex. 44?

47. A solid cylinder $4\frac{1}{2}$ inches high is entirely covered with $45\,\pi$ square inches of paper. Find the radius of the base.

48. Show that the error E in the computed total area of the cylinder of Ex. 47, caused by an error e in measuring the radius, is $E = (2\,e^2 + 21\,e)\pi$. Find e if $E = 50\,\pi$; if $E = -45\,\pi$. Solve for e in terms of E in general.

49. Find the radius of the base of a cone whose slant height is 7 cm. and whose total surface area is $800\,\pi$ sq. cm.

50. A box which is open at the top is to be constructed 20 inches high, with a square base. What are the dimensions of the base, if the surface of the box is 1536 square inches?

51. If an error e, in inches, is made in measuring the side of the base in Ex. 50, show that the error E in the computed area of the box, in square inches, is $E = e^2 + 192\,e$. Find E if $e = 1$; 2; 3; -1; etc. Plot the graph. Find e if $E = 985$.

52. How nearly must the measurement side of the base in Ex. 51 be made in order that the computed area may be correct to within 1 sq. ft.? Is it practicable to do this with a foot rule?

53. A solid block, 10 in. high, is to be twice as long as it is wide. Find its dimensions, if the surface area is 1008 sq. in.

If a body is dropped from a point near the earth's surface, the number of feet it falls in t seconds is given by the relation $s = 16\,t^2$. If instead of being merely *dropped*, the body is *thrown* downward at a

speed of v feet a second, the relation is $s = 16\,t^2 + vt$. If the body is thrown *upward*, the relation is $s = 16\,t^2 - vt$. Notice that in the last instance, since s represents distance *downward*, and the body starts *upward*, s will be negative at first, until $t = v \div 16$.

54. A body is thrown downward with a speed of 50 feet a second from a distance of 225 feet from the earth. When will it strike the ground?

55. When will a body thrown upward with the same speed from the same place as in Ex. 54, strike the ground?

56. When will a body, thrown as in Ex. 55, descend to the level from which it was thrown? Comment on the two answers.

57. When will a stone, thrown downward at a speed of 74 feet a second from a height of one mile, reach the earth?

58. If the error in measuring the time of fall of a body which is *dropped* from a height is e, in seconds, show that the error in the computed value of the distance fallen through is $E = 32\,te + 16\,e^2$, where t is the real time of fall. If $t = 10$, $E = 320\,e + 16\,e^2$. Find E if $e = 1, 2, \frac{1}{2}, \frac{1}{4}, -1, -2$, etc. Plot the graph. Find e if $E = 516$. How carefully must the time be measured in order that the error in the computed distance may be less than 50 ft.? How nearly can the distance be computed with a stop watch which measures to fifths of a second?

59. A body is thrown upward from a height of 1728 feet at a speed of 48 feet a second. When will it reach the earth? When does it reach the level from which it is thrown? How much time elapses between its reaching the level from which it is thrown and its reaching the earth?

With what speed should it be thrown *downward* in order to reach the earth in this time? Comment on the result.

60. Show that the error E in the computed value of the distance fallen through in Ex. 54, caused by an error e in measuring the time of fall, is $E = 130\,e + 16\,e^2$. Plot the graph. Find e if $E = 69$; if $E = -159$. How nearly can the distance be measured by means of a stop watch that reads to fifths of a second?

PART III. PROPERTIES OF QUADRATIC EQUATIONS

117. Given Roots. Factor Theorem. We can now manufacture equations which shall have any roots we wish.

Ex. 1. If $2x^2 - 9x + 4 = 0$,

$$(x-4)(2x-1) = 0.$$

Hence, $\quad x - 4 = 0$, or, $2x - 1 = 0$,

that is, $\quad x = 4$, or, $x = \frac{1}{2}$.

Suppose we wish to *make* an equation with roots 4 and $\frac{1}{2}$, then we write

$$(x-4)(x-\tfrac{1}{2}) = 0,$$

and find the product of these factors. We get

$$x^2 - \tfrac{9}{2}x + 2 = 0,$$

or, multiplying both sides by 2 to clear of fractions, we get

$$2x^2 - 9x + 4 = 0.$$

In general, the roots will be r and s if the equation is

$$(x-r)(x-s) = 0,$$

or, $\quad x^2 - (r+s)x + rs = 0. \quad$ (See § 60.)

To build an equation with two given roots write the product of x less the first root and x less the second root, and set this product equal to zero.

Ex. 2. Given the roots $2 + \sqrt{3}$ and $2 - \sqrt{3}$, find the equation.

The equation is $\quad [x - (2 + \sqrt{3})][x - (2 - \sqrt{3})] = 0.$

Multiplying, we get $\quad x^2 - 4x + 1 = 0,$

which is the desired equation. (See Example 1, p. 207.)

Ex. 3. Given the roots $+3$ and $-\sqrt{2}$, find the equation.

The equation is $\quad (x-3)(x+\sqrt{2}) = 0,$

or, $\quad x^2 - (3-\sqrt{2})x - 3\sqrt{2} = 0.$

The coefficients in this equation are surds; this will always happen if there are surd roots, unless the example is especially selected with roots of the form $a + \sqrt{b}$ and $a - \sqrt{b}$.

The principle of this article may also be stated in a form called the **factor theorem** (See also Appendix, § 5):

If s is a solution of the equation $ax^2 + bx + c = 0$, then $(x - s)$ is a factor of the expression $ax^2 + bx + c$.

118. Relation of Roots to Coefficients. If r and s are the roots of a quadratic equation, the equation is

$$x^2 - (r + s)x + rs = 0,$$

as seen above. Hence,

(1) *The sum of the roots with the sign changed is the coefficient of x in the equation.*

(2) *The product of the roots is the last (or constant) term.*

Notice that the following problem arises from this: Having divided both sides by the coefficient of x^2, any quadratic equation has the form above; to find its roots we must find two numbers whose sum and product we know. Problems of this kind lead to quadratic equations.

Thus, if we know that the sum of two numbers is 6 and their product is 8, these numbers must be the two solutions of the equation $x^2 - 6x + 8 = 0$, *i.e.* 2 and 4. Another method will be found later (See Chapter X, p. 253.)

EXERCISES VII: CHAPTER VIII

Find quadratic equations whose roots are:

1. $2, 3$.

2. $3, 5$.

3. $3, -5$.

4. $-3, 5$.

5. $-3, -5$.

6. $7, -9$.

7. $3, \frac{1}{2}$.

8. $4, -1\frac{1}{3}$.

9. $-2, 2\frac{1}{2}$.

10. $6, -6$.

11. $2\frac{1}{2}, -1\frac{1}{2}$.

12. $-3.4, 7.1$.

13. $3 + \sqrt{5}, 3 - \sqrt{5}$.

14. $-4 - \sqrt{17}, -4 + \sqrt{17}$.

15. $3 - \sqrt{7}, 3 + \sqrt{7}$.

16. $-6 + \sqrt{5}, 2 - \sqrt{5}$.

17. $3 + \sqrt{5}, 2 - \sqrt{5}$.

18. $2 - \sqrt{3}, 3 - \sqrt{3}$.

19. $a + b, a - b$.

20. a, a.

21. $a, -a$.

22. $a + \sqrt{b}, a - \sqrt{b}$.

State the sum and product of the roots in each of the following equations; then check by solving and actually finding the sum and product:

23. $x^2 - 8x + 12 = 0$.

24. $x^2 - 3x = 28$.

25. $x^2 + 7x = 30$.

26. $x^2 + 4x + 3 = 0$.

27. $2x^2 - x - 10 = 0$.

28. $6x^2 + x - 40 = 0$.

29. $x^2 - 2px + p^2 - q^2 = 0$.

30. $4x^2 - 12x + 9 = 0$.

31. $3x^2 + 13x - 10 = 0$.

32. $x^2 - 6x + 1 = 0$.

Find two numbers, given the following data:

	33	34	35	36	37	38	39	40	41
Sum	9	7	-3	-5	10	$-\frac{1}{6}$	$2\frac{1}{2}$	$6p$	0
Product	14	-18	-108	6	3	$-1\frac{5}{9}$	$-12\frac{1}{2}$	$9p^2$	$-4q^2$

119. Factoring Quadratic Expressions. Another result of the work just done is a new method of factoring *quadratic expressions*, *i.e.* expressions of the form

$$ax^2 + bx + c,$$

such as occur on the left side of a quadratic equation when the right side has been reduced to zero. (For elementary method see §§ 60, 61, pp. 95–99.)

Ex. 1. Factor the quadratic expression $2x^2 - 9x + 4$.

$$2x^2 - 9x + 4.$$

Let us first solve the quadratic equation

$$2x^2 - 9x + 4 = 0,$$

which is formed by setting the given *expression* equal to zero.

The solutions are,

$$x = 4 \text{ and } x = \tfrac{1}{2} \text{ (see p. 203).}$$

Hence, $(x - 4)$ and $(x - \frac{1}{2})$ are each factors of $2x^2 - 9x + 4$ (see § 117). The other factor is 2, since we know we must produce $2x^2$ as the first term. It follows that

$$2x^2 - 9x + 4 = 2(x - 4)(x - \tfrac{1}{2}),$$

or,

$$2x^2 - 9x + 4 = (x - 4)(2x - 1).$$

The desired factors are therefore $x - 4$ and $2x - 1$.

Ex. 2. Factor $x^2 - 4x + 1$.

The solutions of $x^2 - 4x + 1 = 0$ are $x = 2 + \sqrt{3}$ and $x = 2 - \sqrt{3}$;

hence, $$x^2 - 4x + 1 = [x - (2 + \sqrt{3})][x - (2 - \sqrt{3})].$$

The desired factors are $(x - 2 - \sqrt{3})$ and $(x - 2 + \sqrt{3})$.

It is to be noted that the *coefficients* in such factors may be irrational; thus, $2 + \sqrt{3}$ is irrational. The factors are strictly *polynomials*, however, since the important letters do not occur irrationally, but only in simple powers.

Ex. 3. Factor $x^2 - 2$.

The solutions of $x^2 - 2 = 0$ are $x = +\sqrt{2}$ and $x = -\sqrt{2}$;

hence, $$x^2 - 2 = [x - \sqrt{2}][x - (-\sqrt{2})].$$

The desired factors are $(x - \sqrt{2})$ and $(x + \sqrt{2})$.

Ex. 4. Factor $x^2 - 4x + 4$.

The only solution of $x^2 - 4x + 4 = 0$ is $x = 2$. Hence, $x^2 - 4x + 4$ is a *perfect square* $(x - 2)^2$, as is seen on inspection. *This will always happen if* $b^2 - 4ac = 0$, which is therefore the condition that $ax^2 + bx + c$ shall be a perfect square.

Ex. 5. Factor $ax^2 + bx + c$.

The solutions of $ax^2 + bx + c = 0$ are $x = \dfrac{-b}{2a} \pm \dfrac{\sqrt{b^2 - 4ac}}{2a}$;

hence,

$$ax^2 + bx + c = a\left[x - \left(\frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right)\right]\left[x - \left(\frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right)\right]$$

where the factor a on the outside is chosen so that the term in x^2 will be ax^2.

The result is meaningless if $b^2 - 4ac < 0$, for in that case the expressed radical $\sqrt{b^2 - 4ac}$ has no meaning at present. It is helpful to notice the following scheme of results:

Discriminant ($b^2 - 4ac$)	Solutions of $ax^2 + bx + c = 0$	Factors $ax^2 + bx + c$
positive	real unequal roots (two solutions)	real and unequal (factorable)
zero	equal roots (only one solution)	equal: perfect square
negative	imaginary roots (no solution)	imaginary (factoring impossible)

The principle to be used here is nothing more than that stated in § 117: *If r is a root of a quadratic equation whose right side is zero, $(x - r)$ is a factor of the left side.*

120. Solution by Factoring. As in Ex. 5, p. 226, we may factor any quadratic expression, the factors being meaningless for the present if $b^2 - 4ac < 0$. We may do this same work independently as follows:

$$ax^2 + bx + c = \frac{(4a^2x^2 + 4abx + b^2) - (b^2 - 4ac)}{4a},$$

as is seen by reducing the right side. Or,

$$ax^2 + bx + c = \frac{(2ax + b)^2 - (\sqrt{b^2 - 4ac})^2}{4a},$$

in which form the factors can be seen. (See also Ex. 5, § 119.)

Ex. 1. $x^2 - 4x + 1 = (x^2 - 4x + 4) - (4 - 1) = (x - 2)^2 - (\sqrt{3})^2$
$= (x - 2 - \sqrt{3})(x - 2 + \sqrt{3}).$

As in this example, the student should not try to remember the formulas above, but he should rather try to complete the square as in § 206.

It is interesting to notice that a quadratic equation may always be *solved* by this method *of factoring:*

Since $\qquad x^2 - 4x + 1 = (x - 2 + \sqrt{3})(x - 2 - \sqrt{3}),$

as we just saw, the quadratic equation

$$x^2 - 4x + 1 = 0$$

may be written $(x - 2 + \sqrt{3})(x - 2 - \sqrt{3}) = 0.$

Hence, *either* $(x - 2 + \sqrt{3}) = 0$, or, $(x - 2 - \sqrt{3}) = 0.$

(See § 66, p. 107.)

And $x = 2 - \sqrt{3}$, or, $x = 2 + \sqrt{3}.$

This method, while interesting, is not often used in difficult examples, because it is more complicated than the methods already given. In easy examples, however, where the factors may be found readily by inspection, the method by factoring is by far the quickest. We have used it above very often; see pp. 103–115, 151, 203.

EXERCISES VIII: CHAPTER VIII

In the following exercises, note the values of a, b, c; determine $b^2 - 4ac$ and classify the expressions according as $b^2 - 4ac >$ or $=$ or < 0. When $b^2 - 4ac \geq 0$, find the roots of the corresponding equations, and thus factor the expressions:

1. $x^2 - 9x + 14.$

2. $p^2 - 7p + 13.$

3. $6m - m^2 - 9.$

4. $3t^2 + t - 2.$

5. $z^2 + z + 1.$

6. $z^2 - z + 1.$

7. $z^2 + z - 1.$

8. $3v^2 - 30v + 75.$

9. $4x^2 - 20x + 25.$

10. $7u^2 - 12u + 4.$

11. $5u^2 - 3u + 2.$

12. $5x^2 - 3x - 2.$

13. $6c^2 - 11c + 3.$

14. $t^2 - 12t.$

15. $7r^2 - 9r + 4.$

16. $2s^2 - 10s + 12\frac{1}{2}.$

17. $11z^2 + 2z - 40.$

18. $9t^2 + 42t + 49.$

19. $13t^2 + 5t.$

20. $3t^2 - 2t + 105.$

21. $x^2 + .3x + .02.$

First factor each of the following expressions, *then* find the roots of the corresponding equation:

22. $x^2 - 4.$

23. $4a^2 - 9.$

24. $25m^2 - 1.$

25. $g^2 - 17g + 30.$

26. $x^2 + 8x - 33.$

27. $x^2 + 5x.$

28. $10y^2 - y.$

29. $4x^2.$

30. $6z^2 + z - 35.$

31. $4m^2 - 2m - 12.$

32. $15r^2 - 34r + 15.$

33. $x^2 - .3x + .02.$

121. Literal Coefficients. The student has solved some exercises in which letters occurred in the coefficients. We shall now solve such equations systematically.

Ex. 1. Given $x^2 - 2mx + (m^2 - 1) = 0$.

Transpose $m^2 - 1$ and add m^2 to both sides:

$$x^2 - 2mx + m^2 = -(m^2 - 1) + m^2,$$

or, $(x - m)^2 = 1$, or, $x - m = \pm 1$, or, $x = m \pm 1$.

This solution holds for any value of the letter m. (Check it.)

Ex. 2. Given $x^2 - 12xy + 4y^2 = 0$.

Solving for x, we find $x^2 - 12xy + 36y^2 = 32y^2$,

or, $$x - 6y = \pm\sqrt{32y^2},$$

or, $$x = 6y \pm \sqrt{32y^2} = y(6 \pm 4\sqrt{2}). \quad \text{(Check it.)}$$

Solving for y, we find $4y^2 - 12xy + 9x^2 = 8x^2$,

or, $$2y - 3x = \pm\sqrt{8x^2},$$

or, $$y = \frac{3x}{2} \pm \frac{1}{2}\sqrt{8x^2} = \frac{x}{4}(6 \pm 4\sqrt{2}). \quad \text{(Check it.)}$$

Thus, we may solve the same equation for any letter in it.

*NOTE ON THE DISCRIMINANT. The given equation has equal roots if the discriminant is zero.

Let d stand for discriminant; then the roots of the given equation are real and unequal if $d > 0$; imaginary if $d < 0$; equal if $d = 0$.

Ex. 3. Given $x^2 + kx + (3 + k) = 0$.

Comparing with $ax^2 + bx + c = 0$,

we find $a = 1$, $b = k$, $c = 3 + k$; hence, the discriminant

$$b^2 - 4ac = k^2 - 4(3 + k) = k^2 - 4k - 12.$$

Let us try several values of k. If $k = 0$, $d = -12$, and the roots are imaginary; in fact, the equation is

$$x^2 + 0 \cdot x + (3 + 0) = 0,$$

or, $$x^2 + 3 = 0,$$

or, $x = \pm\sqrt{-3}$, which is imaginary.

* This work may be omitted unless especially desired; in any case it should not be attempted until the student is quite proficient.

Draw the figure for $k = 0$, and show that it does not cut the main horizontal line.

If $k = 1$, $d = 1 - 4 - 12 = -15$; roots imaginary.

What is the original equation in this case ($k = 1$)?

Solve it. Are the solutions imaginary? Draw the figure.

If $k = 6$, $d = 36 - 4 \cdot 6 - 12 = 0$; roots equal.

What is the original equation in this case ($k = 6$)?

Solve it. Are the roots equal (*i.e.* only one solution). Draw.

If $k = 10$, $d = 100 - 4 \cdot 10 - 12 = 48$; roots real and unequal.

What is the original equation in this case ($k = 10$)?

Solve it. Are the roots real and unequal? Draw.

Trying several other values of k, we make this table:

k	0	1	2	3	etc.	5	6	7	etc.	-1	-2	-3	etc.
d	-12	-15				-7	0	9		-7	0	9	
roots	imag.	imag.				imag.	equal	real		imag.	equal	imag.	

[Let the student fill in the blank spaces and extend this table to $k = +10$ and backward to $k = -10$.]

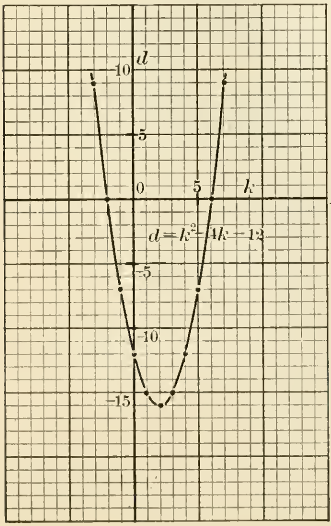


Fig. 40.

We may draw these values of k and d as shown in Fig. 40. From this it is clear that:

$d = 0$ only when $k = -2$ and when $k = 6$ (*i.e.* the original equation has equal roots *only when* $k = +6$ or -2).

$d < 0$ when k has any value between -2 and $+6$ (*i.e.* the original equation has imaginary roots when and only when k is a number between -2 and $+6$).

$d > 0$ when k is less than -2, also when k is greater than $+6$ (*i.e.* the original equation has real and unequal roots when $k < -2$, and when $k > +6$).

These results may also be *found by factoring* the discriminant: $d = k^2 - 4k - 12 = (k + 2)(k - 6)$.

EXERCISES IX: CHAPTER VIII

Solve the following equations for the letters indicated:

1. $x^2 - 5\,ax = -6\,a^2$. (First for x; then for a.)

2. $x^2 - 6\,ax^2 = -5\,a^2$. (x, a.)

3. $m^2 - 6\,mx = r^2 - 9\,x^2$. ($m$, x.)

4. $mr^2 - (m+n)rs + ns^2 = 0$. ($r$, s.)

5. $2\,z^2 - 3\,mz - 14\,m^2 = 0$. ($z$, m.)

6. $3\,a^2 + 8\,ab - 3\,b^2$. (a, b.)

7. $p^2 - 2\,pm - (2\,m + 1) = 0$. ($p$.)

8. $y^2 - 6\,ay - 6\,a - 1 = 0$. ($y$.)

9. $t^2 - mt + m = 1$. (t.)

10. $x^2 + 6\,y^2 + 6\,z^2 + 12\,yz - 5\,zx - 5\,xy = 0$. ($x$, y, z.)

Find for what values of k the roots of the following equations will be real and unequal, real and equal, or imaginary; in the first two cases, solve the equations.

[Omit these until very proficient.]

11. $x^2 - kx + k = 0$.

12. $x^2 - (k+2)x + 2(k+2) = 0$.

13. $x^2 + 2(k-3)x + (k+3) = 0$.

14. $kx^2 + (k+5)x + (k+5) = 0$.

15. $\dfrac{x^2}{k+3} + \dfrac{x}{k} + \dfrac{1}{k+3} = 0$.

16. The area of a square field measured in square rods is equal to the length of its whole perimeter measured in rods less twice a given number k. For what values of k is the problem possible? impossible? For what values are there two answers for the size of the field? When is there just one answer?

REVIEW EXERCISES X: CHAPTER VIII

Solve the following equations:

1. $z^2 - 5z - 300 = 0.$
2. $t^2 - 3t - 108 = 0.$
3. $x^2 + 34x - 800 = 0.$
4. $x^2 - 29x + 168 = 0.$
5. $p^2 - 6p - 247 = 0.$
6. $6x^2 - 7x - 20 = 0.$
7. $15r^2 - 22r - 91 = 0.$
8. $6q^2 + 5q - 781 = 0.$
9. $14y^2 - 3y - 270 = 0.$
10. $8s^2 - 18s - 425 = 0.$
11. $12x^2 - x - 20 = 0.$
12. $18a^2 - 79a - 3100 = 0.$
13. $4k^2 + 7k - 147 = 0.$
14. $13k^2 + k - 120 = 0.$
15. $28x^2 - 3x - 135 = 0.$
16. $2y^2 - 5y - 88 = 0.$
17. $18y^2 - 31y + 6 = 0.$
18. $6x^2 + 7x - 49 = 0.$
19. $12x^2 + 5x - 72 = 0.$
20. $35k^2 - k - 6 = 0.$

Factor the following expressions by first solving the corresponding equations:

21. $6x^2 - x - 77.$
22. $24p^2 - 94p - 63.$
23. $6k^2 - 7k - 33.$
24. $y^2 - 66y - 675.$
25. $12k^2 + k - 130.$
26. $21r^2 + 13r + 20.$
27. $6z^2 - 7z - 24.$
28. $50r^2 - 5r - 36.$
29. $21m^2 - 20m - 96.$
30. $24x^2 + 19x - 35.$

Form equations whose roots shall be:

31. $6, -8.$
32. $\frac{1}{2}, \frac{1}{3}.$
33. $\frac{2}{3}, \frac{-3}{4}.$
34. $5, \frac{-1}{2}.$
35. $\frac{-1}{2}, \frac{-2}{3}.$
36. $3, \frac{1}{3}.$
37. $7, \frac{-2}{3}.$
38. $-5, 1\frac{1}{2}.$
39. $a - 3b, 2a - b.$
40. $\frac{a}{b}, \frac{b}{a}.$

Find the discriminant of each of the following equations; determine the character of the roots, and solve if the roots are real:

41. $14x^2 + 29x - 15 = 0.$

42. $81z^2 - 198z + 121 = 0.$

43. $8t^2 - 13t + 6 = 0.$

44. $5y^2 + 16y + 11 = 0.$

45. $3x^2 - 11x + 11 = 0.$

46. $6x^2 - 11x + 4 = 0.$

47. $x^2 - (k+3)x + k^2 = 0.$

48. $t^2 - (k+3)t - k = 0.$

49. $(k-1)r^2 + 2kr + (k+1) = 0.$

50. $(k+1)x^2 - (3k+1)x + (k+1) = 0.$

51. The sum of two numbers is 16; the sum of their squares is 130. What are the numbers?

52. The sum of two numbers is s; the sum of their squares is s. Show that the quadratic equation found in order to determine one number has for its other root the other number.

Sketch of Solution. Let n_1 and n_2 denote the two numbers. Then $n_1 + n_2 = s$ or $n_1 = s - n_2$; since $n_1^2 + n_2^2 = s$, the equation for either n is of the form $n^2 + (s-n)^2 = s$; hence the two n's are precisely the two roots of this equation.

53. The sum of two numbers is s; if each of the numbers is divided into A, the sum of the quotients is s. Show that the same quadratic equation has both numbers for its roots.

54. Four consecutive integers have as the sum of their squares 54. What are the integers?

55. A and B together can fold 1000 circulars in an hour. It is observed that when each works separately at 1000 circulars, A finishes 50 minutes before B. How many circulars can each man fold in an hour?

56. Two tanks have capacities in the ratio 3 to 4. When unequal pipes are attached, the small tank is filled in 2 hours less time than the large tank. The two tanks are then connected, and both pipes used to empty them. This process requires 2 hours 48 minutes. How long would be required for each pipe alone to empty each tank?

[Suggestion. The student may introduce the idea of a "unit tank" whose capacity is one third that of the smaller tank; use as the principal unknown the time, t, required to fill the small tank with its pipe, and express the capacity of each pipe (c_1 and c_2), *i.e.* the amount each can carry in one hour, in terms of t.]

57. An open box 8 cm. high, whose base is a rectangle with sides in the ratio 3 to 4, is to have a surface area of 640 sq. cm. What are the dimensions of the base?

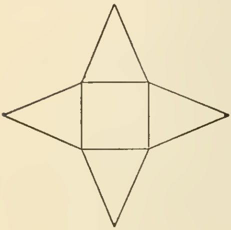


Fig. 41.

58. A regular pyramid on a square base is to be made by folding a figure like that shown. The altitude of the triangular sides is to be 6 in. The surface area of the pyramid is *not to exceed* 100 sq. in. What must be the side of the base to obtain precisely that surface area? (Solve first graphically and then directly from the equation.)

59. What three consecutive integers can measure the sides of a right triangle?

60. Show that if one perpendicular side and the hypotenuse of a right triangle are measured by consecutive integers, the square of the other perpendicular side must be measured by an odd number. Choose for the square of this side successively 9, 25, 49. In each case what are the other two sides?

61. A body is thrown upward at a height of 500 feet with a speed of 30 feet a second. When will it reach the earth? (Solve first graphically, then directly from the equation.) See p. 221.

A body thrown horizontally into the air at a height of h feet with a speed v feet a second, follows a path thus described: in t seconds the horizontal flight, x feet, and the distance from the earth, y feet, of the body are given by $x = vt$, $y = h - 16\,t^2$. A body thrown into the air from the earth's surface at an angle of 45° with a speed of v follows the path $x = \frac{vt}{\sqrt{2}}$, $y = \frac{vt}{\sqrt{2}} - 16\,t^2$.

62. What is the relation between x and y, *i.e. the equation of the path of the body*, in each of the above cases?

63. A stone is thrown horizontally from a cliff 400 feet high at a speed of 50 feet a second. When will the stone strike the earth, and how far from the foot of the cliff?

64. When will the horizontal distance traveled by the stone be equal to its height? What will this distance be? (Solve graphically first.)

65. A body is thrown into the air at an angle of 45° with a speed of 37 ft. a second. At what horizontal distance will it be 5 ft. from the ground? How long has it then traveled?

66. At what horizontal distance will the body be $\frac{37^2}{128}$ feet from the ground?

Consider the graph carefully in connection with Exs. 65, 66.

[The following exercises are intended only for use upon a review of the whole book and are not to be solved until the student has completed the study of Chapter XII.]

Solve the equations:

67. $x^4 + 6x^2 = 40.$

68. $\sqrt{x} + 5\sqrt[4]{x} = 14.$

69. $(x^2+1)^2 + 8(x^2+1) = 180.$

70. $\sqrt{2x+3} + 6x = 71.$

71. $\sqrt{x+7} + \dfrac{1}{\sqrt{x+7}} = 3\frac{11}{14}.$

72. $x^2 - 3x - 3\sqrt{x^2 - 3x + 7} = 3$

73. $\sqrt[6]{x} + \sqrt[3]{x} = 2.$

74. $2x^2 + 7x + 6 - 3\sqrt{8x^2 + 28x - 11} = 0.$

75. $2\left(x + \dfrac{1}{x}\right)^2 - 9\left(x + \dfrac{1}{x}\right) + 10 = 0.$

SUMMARY OF CHAPTER VIII: QUADRATIC EQUATIONS, pp. 203–235

CHAPTER IX

VARIATION: INDETERMINATE EQUATIONS

122. Simple Variation. We have discussed before quantities that vary. Thus, the cost of an amount of butter varies with the number of pounds bought. (See p. 23.)

Whenever the quotient of two varying quantities y and x is a constant k,

$$\frac{y}{x} = k, \text{ or, } y = kx,$$

the variables y and x are said to be **in proportion** (see pp. 25, 140, etc.), by which we mean that any pair of values of y and x form a proportion with any other pair. We also say in this case that **y varies directly as x.**

We have seen, p. 141, that the corresponding figure is a *straight line through the starting point.*

The following are therefore *synonymous:*

(1) *y is proportional to x.*

(2) *y varies directly as x.*

(3) *The quotient $y \div x$ is constant; or, $y = kx$.*

(4) *The graph for y and x is a straight line through the starting point.*

Instead of the quotient $y \div x$, we may speak of the ratio of y to x and write it $y : x$; and we may say that the ratio $y : x$ is a constant. This constant, which is called k above, is called the **ratio of proportionality.**

Thus, if butter costs 30¢ per pound (see p. 23), the ratio $c : p$ or $\frac{c}{p}$, where c denotes the cost in cents and p denotes the number of

pounds, is always 30. The ratio of proportionality is 30. Any pair of values of c and p, say c_1 and p_1, give the same ratio as any other pair, c_2 and p_2:

$$\frac{c_1}{p_1} = \frac{c_2}{p_2}\ (=30),\ \text{or,}\ \frac{c_1}{c_2} = \frac{p_1}{p_2}.\quad \text{(See VI, (1), p. 138.)}$$

If one varying quantity, z, varies directly as the product, $x \times y$, of two other quantities x and y, then z evidently varies directly as x when y is constant, and as y when x is constant. For if $z = k \cdot x \cdot y$, where k is a constant, then $z = (k \cdot x) \cdot y$, whence z is a constant times y if x is a constant. Likewise $z = (k \cdot y) \cdot x$, whence z is a constant times x if y is a constant.

Thus, the area, A, of a rectangle is given by the formula

$$A = b \times h,$$

where b is the base and h is the height of the rectangle, measured in feet or in any other unit of length. If b is constant, A varies as h, *i.e.* the areas of rectangles with equal bases are to each other as the heights. Likewise A varies as b if h is constant, *i.e.* the areas of rectangles of the same height are to each other as the bases.

The reverse statement is also true: if z varies as x when y is constant, and as y when x is constant, then z varies as the product $x \times y$.

Thus, if we know that the area, A, of a rectangle varies as h (the height) when b (the base) is a constant, and as b when h is a constant, we may conclude that A varies as $b \times h$. A formal proof of this last form of statement is deferred.

123. Linear Variation. We have also seen that a linear equation (or equation of the first degree) of the form

(1) $$y = ax + b,$$

where a and b are constants, is represented by a straight-line graph. (See pp. 26, 143.) This kind of relation is often called **linear variation**, and we say that y is a **linear function** of x. Examples of this occur on pp. 25, 143.

Ex. 1. As another example consider the equation

$$2x - 3y = 6.$$

Solve for y: $$y = \tfrac{2}{3}x - 2,$$

which is of this same type, *i.e.* the curve is a straight line. To draw it we need only plot *two* points (see footnote, p. 144). For example, $(x = 0, y = -2)$ and $(y = 0, x = 3)$ as can be seen also from the original equation $2x - 3y = 6$. The graph is as shown in Fig. 42.

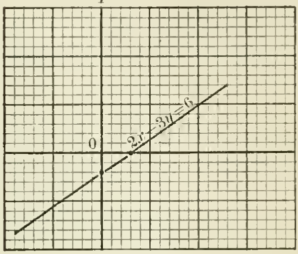


FIG. 42.

Any equation of the form

(2) $$Ax + By + C = 0$$

can be reduced to the type (1) provided $B \neq 0$, as in the preceding example. If $B = 0$, the equation

$$Ax + C = 0, \text{ or, } x = \frac{-C}{A}$$

does not contain y. Hence, $x =$ *the same thing* for every possible y, and the curve is a *vertical* straight line.

In all cases, without exception, an equation of the first degree is represented in a figure by a straight line.

124. Inverse Variation. It may happen that one varying quantity *increases* as another *decreases*, in such a way that *their product is constant.*

Thus, if a train goes 20 miles per hour on a trip 600 miles long, the time taken is 30 hours; if it goes 25 miles per hour, the time taken is 24 hours; if it goes 30 miles per hour, the time taken is 20 hours. Notice the product speed $\times$ time $=$ constant $=$ total distance:

$$s \cdot t = d,$$

where s stands for *speed*, t for *time*, d for *distance*. A table follows:

s	1	2	5	10	15	20	30	40	50	60	etc.	-10	-30	etc.
t	600	300	120	60	40	30	20	15	12	10	etc.	-60	-20	etc.
d	600	600	600	600	600	600	600	600	600	600		600	600	

Here d is a constant, 600 (in miles). But s and t vary. The graph of the relation between s and t is shown in Fig. 43. Negative values of s and t correspond to backward motion.

A relation between two varying quantities — say y and x — such that

$$\boldsymbol{x \cdot y = k} \left(\text{or } x = \frac{k}{y}, \text{ or } y = \frac{k}{x}\right),$$

where k is a constant, is called **inverse variation**, and we say that ***y* varies inversely as *x***.

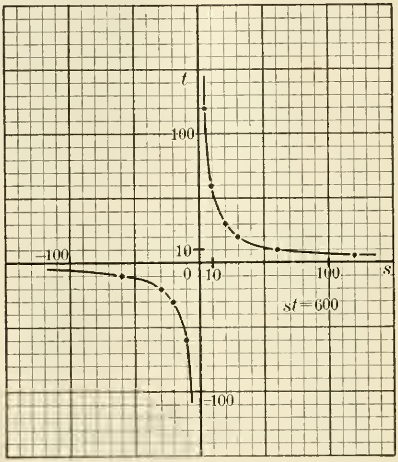


FIG. 43.

Curves shaped like the above, *i.e.* curves corresponding to an equation of the form $xy = k$, we shall call *inverse variation curves.* Similarly, we notice that *direct variation curves are straight lines.*

EXERCISES I: CHAPTER IX

Draw graphs to represent the following relations:

1. $y = 16x$.

2. $y = -12x + 7$.

3. $x = 3y - 5$.

4. $2x + 5y = 9$.

5. $3x + 7y = -18$.

6. $y = 5(x - 2)$.

7. $xy = 1$.

8. $xy = 100$.

9. $xy = 72$.

10. $xy = 540$.

11. $xy = -1$.

12. $xy = -100$.

13. $xy = -600$.

14. $x(y-3) = 100$.

15. $(x-2)y = 75$.

16. $(x-3)(y-2) = 500$.

17. Suppose y varies directly as x and the ratio of proportionality is 2; write the equation; draw a graph; find y when $x = 1$, when $x = 2$.

18. Suppose y varies directly as x and $x = 2$ when $y = 10$; write the equation; draw a graph; find y when $x = 3$.

19. Suppose y varies inversely as x and $x = 1$ when $y = 2$; write the equation; draw the graph; find y when $x = 2$.

20. Suppose y varies as the square of x and $x = 1$ when $y = 4$; write the equation; draw the graph; find y when $x = 2$, when $x = 3$.

21. The volume, V, of a box whose height is 4 ft. is $V = 4\,wl$, where w is the width and l is the length (in feet), and where V is the volume (in cubic feet). Show that V varies as w when l is constant, and as l when w is constant.

22. Show that the area of an open circular cylinder varies as the height (h) when the radius (r) of the base is a constant, and as r when h is constant. (See Tables.)

23. Find from the Tables all geometrical figures whose areas, or volumes, vary directly as certain of their dimensions, and express each of these both by formulas and in words.

24. The cost of any number of pounds of butter varies as the number of pounds. If one pound costs 30 cents, express the cost of any number of pounds; draw the graph; what is the cost of 5 pounds?

25. For a certain mass of gas, it is observed that pressure times volume is equal to 120,000. Plot the graph. (See p. 218.)

26. Indicate by a picture the relation between base and altitude of a triangle of constant area.

27. If a certain stretched wire is caused to vibrate, then the number n of vibrations a second, the radius r of the wire in centimeters, and its length l in centimeters are observed to satisfy the relation $nrl = 16000$.

For a wire 50 cm. long, plot the relation between n and r.

28. Express by a figure the radius and length of all wires sounding F of the middle register ($n = 320$ per second).

29. The surface area of a cylindrical ring is 330 sq. in. Express by an equation, and plot a figure for all possible values of the radius and height of the ring. (Take $\pi = \frac{22}{7}$.)

30. The relation between the mass, volume, and density of a body is $m = vd$.

Choose a suitable constant value for the density, and plot the relation between volume and mass.

31. Choose a suitable constant value for the mass, and plot the relation between volume and density.

32. Express graphically the relation between the total surface area and the slant height of a right circular cone whose base has a radius of 7 inches.

125. Indeterminate Equations. The examples above are all examples of **indeterminate equations,** *i.e.* equations in which each of the unknown letters has not one fixed value, but rather an indefinite number of possible values.

Sometimes a problem is of such a nature that only a few possibilities remain; we may then fix definite values for the letters.

Ex. 1. A carpenter has boards 8 in. wide and others 6 in. wide. How many of each may he take to make a walk 50 in. wide, the boards being laid lengthwise?

Let x = number of 8-inch boards, and y = number of 6-inch boards.

Then $$8x + 6y = \text{width},$$

or, $$8x + 6y = 50.$$

Let us now try various numbers. If $x = 0$, $y = \frac{25}{3} = 8\frac{1}{3}$. ($A$ in Fig.) If $y = 0$, $x = \frac{25}{4} = 6\frac{1}{4}$. ($B$ in Fig.)

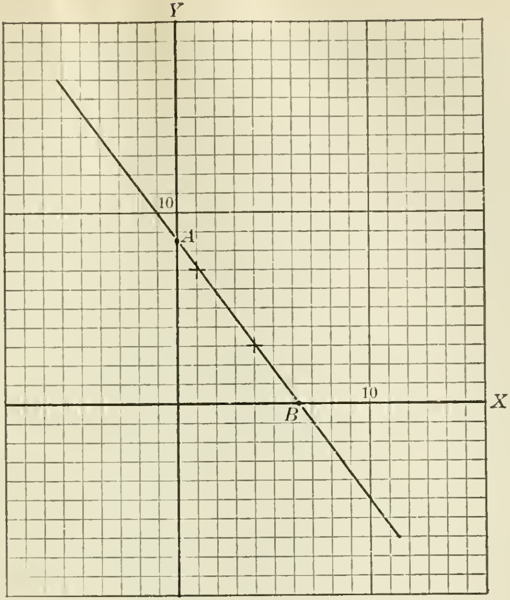


FIG 44.

The figure is a straight line, for the equation is of the first degree. Hence, the possible values of x and y correspond to the points on the straight line joining A and B.

Now the carpenter must choose an *integral* number of each sort of boards. The values of x and y must then be integers, *i.e.* the corresponding points lie at the intersections of the square paper. We notice three points that *may* serve:

$$\begin{Bmatrix} x = 1, \\ y = 7, \end{Bmatrix} \qquad \begin{Bmatrix} x = 2, \\ y = 6, \end{Bmatrix} \qquad \begin{Bmatrix} x = 4, \\ y = 3, \end{Bmatrix}$$

for each of these lies at least very close to the straight line. Trying these, we find $x = 2$, $y = 6$ gives a total width $2 \cdot 8 + 6 \cdot 6 = 52$ in. This is wrong.

The point $(x = 1,\ y = 7)$ gives $1 \cdot 8 + 7 \cdot 6 = 50$. This is correct. Likewise $(x = 4,\ y = 3)$ is correct. The carpenter may, therefore, choose 1 board 8 in. wide and 7 boards 6 in. wide, or 4 boards 8 in. wide and 3 boards 6 in. wide. If he has more 8-inch boards than 6-inch boards, he should clearly choose the latter scheme.

Ex. 2. A man agrees to exchange young hogs at \$3.00 each for sheep at \$5.00 each. How many sheep and hogs may be traded practically?

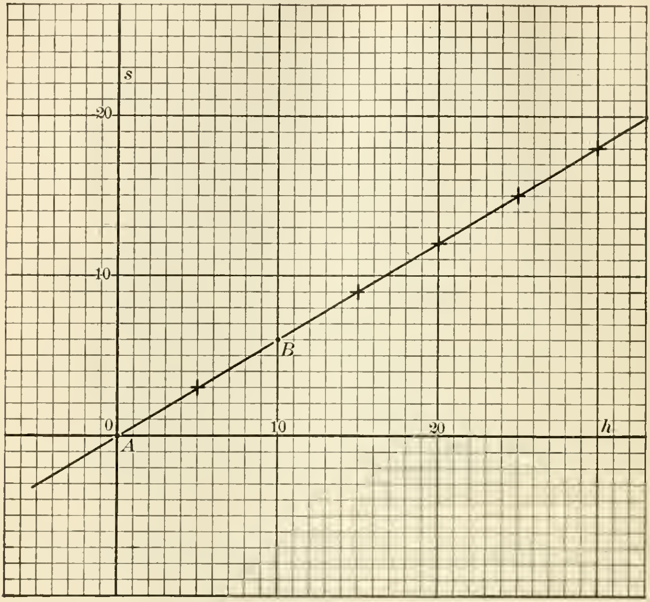


Fig. 45.

Let $h =$ the number of hogs, and $s =$ the number of sheep.

Then $$3h = 5s,$$

or, $$3h - 5s = 0.$$

If $h = 0$, $s = 0$. If $h = 10$, $s = 6$.

The graph is surely a straight line since the equation $3h - 5s = 0$, is linear. Drawing a figure, the possibilities are seen to be only these:

Hogs	0	5	10	15	20	etc.
Sheep	0	3	6	9	12	etc.

Ex. 3. A frame (box without a bottom or top) 2 ft. high is to be constructed out of a board 1 ft. wide and 24 ft. long. Find how to cut the board to make the largest box.

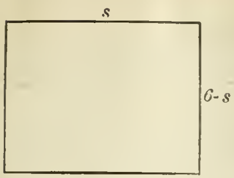


Fig. 46.

Let s = length of one side,

then $6 - s$ = length of other side.

Then area of bottom $= s(6 - s)$.

The largest box is that which has the largest bottom. We wish to find when $b = s(6 - s)$ is largest, where b means the area of the bottom in square feet. If $s = 0, b = 0$; if $s = 1, b = 5$; if $s = 2, b = 8$; etc. Proceeding in this way, we find a table as follows:

s	0	1	2	3	4	5	6	etc.
b	0	5	8	9	8	5	0	etc.

From the figure, the highest value of b is seen to be where $s = 3$, at least approximately. Hence, the largest box will result by taking the side approximately 3 ft. The shape of the bottom will then be a square; and, as a matter of fact, this is the shape that will give the greatest volume to the box.

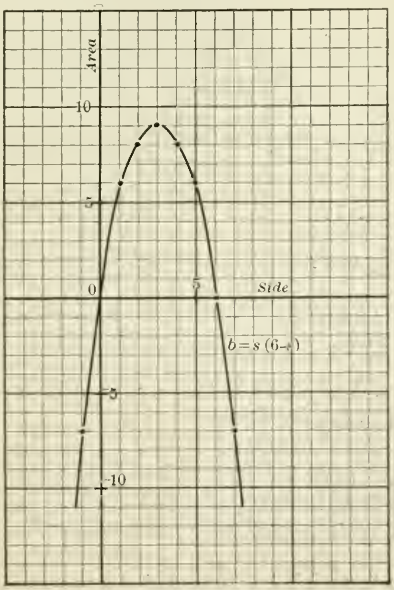


Fig. 47.

EXERCISES II: CHAPTER IX

1. Find all positive integral solutions of the equation $2x + 7y = 50$.

2. Find some integral solutions of the equation $3x - 5y = 7$.

3. Find all integral solutions of the equation $\frac{x}{3} + \frac{y}{4} = 3$.

4. Find all even positive integral solutions of the equation $5x + 7y = 96$.

5. Find all positive integral solutions of the equation $5x + 7y = 99$, such that x is odd and y even.

6. Show that the equation $5x + 7y = 99$ is satisfied by all values of x and y given by the formulas $x = 17 - 7s$, $y = 5s + 2$. For what values of s are the results of Ex. 5 obtained?

7. Show that the equation $5x - 6y = 7$ is satisfied by the formulas $x = 6s - 1$, $y = 5s - 2$. Give several values to s, and determine values of x and y. Also solve graphically.

Find the least positive integral solution of the following equations:

8. $7x - 8y = 17$.

9. $12x + 5y = 49$.

10. $3x - 14y = 1$.

11. $49x - 53y = 17$.

12. How can I pay a debt of 60 cents in quarters and dimes? Give all selections.

13. How can I pay \$57 in two- and five-dollar bills? Which method uses the fewest bills?

14. A farmer pays a debt of \$4.50 by giving chickens valued at 30 cents each and turkeys valued at 80 cents each in trade. How many of each sort of fowl are needed?

15. I desire to weigh 100 pounds by means of 3- and 11-pound weights placed in the same pan. How can this be done?

16. How can this be done if the two kinds of weights are to be placed in opposite pans, the 3-pound weights being in excess?

17. How can this be done, the 11-pound weights being in excess?

18. A walks 3 hours to reach a neighboring town; B, starting from the same place, walks 5 hours and is still a mile from the town. If each man walks in an hour an integral number of miles, how fast do the men walk, and how far distant is the town?

Note that one condition of this problem is the limitation of possible speed of human walking.

19. Find a number that leaves a remainder of 11 on division by 16, and of 4 on division by 11.

Show also that every number of the form $176\,t + 59$, where t is a positive integer, has this property.

20. What is the perimeter y of a rectangle having one side x and area A? Plot the relation between y and x for several values of A. On each curve note the value of x, which makes y a minimum. What rectangle of given area has the minimum perimeter?

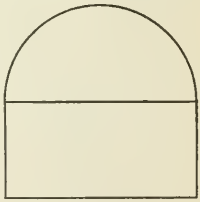


FIG. 48.

21. A window having the shape of a rectangle surmounted by a semicircle has a total perimeter of 200 inches. What dimensions should it have to admit the greatest amount of light? (Take $\pi = \frac{22}{7}$.) What is then the area?

126. Other Cases. Many other forms of indeterminate equations arise. Some figures have been drawn on pp. 30, 182, 204, to illustrate these, particularly in connection with quadratic equations.

In general, given an equation containing two varying quantities, we give several values to one of them, find the corre-

sponding values of the other, and plot in a figure the points that correspond to each pair of values. Finally we connect these points by a smooth curve.

In case there is any doubt as to how the curve should be drawn, we simply plot more points, as above, until the curve is fairly outlined by them.

Among interesting curves are the following:

Ex. 1. $y = kx^2$, where k is constant.

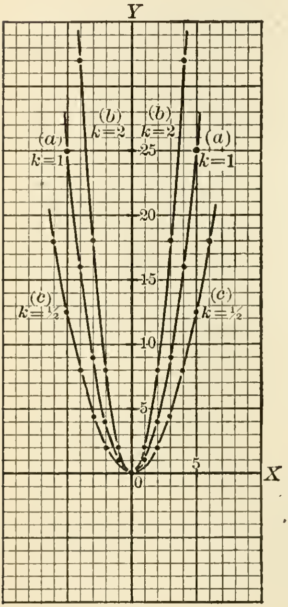


FIG. 49.

This arises often practically. Thus, the price of painting a square surface varies as the square of one side. $p = k \cdot s^2$ where k is the price (in dollars) for painting 1 sq. yd., and where p means the total price (in dollars), and s the length of one side in yards.

The length l (in feet) of a pendulum varies as the square of the time t of vibration (in seconds):

$$l = k \cdot t^2 \text{ where } k = \frac{32.16}{\pi^2} = 3.26 \text{ (nearly).}$$

The distance d (in feet) traversed by a body falling from rest varies as the square of the time t (in seconds) it falls:

$$d = k \cdot t^2 \quad \text{where} \quad k = 16.08.$$

The equations of the form

$$l = ax^2 + bx + c$$

considered in the previous chapter are of the same type, but are somewhat more complicated than this example.

(*a*) Let us draw $y = x^2$, *i.e.* the case in which $k = 1$.

Making a table as before, we find:

x	0	1	2	3	4	5	6	7	etc.	-1	-2	-3	-4	etc.
y	0	1	4	9	16					1	4			etc.

[Let the student fill in the blanks.]

The graph is as drawn in Fig. 49, the curve marked (a).

(b) If $y = 2x^2$, we find

x	0	1	2	3	4	5	6	etc.	-1	-2	-3	-4	etc.
y	0	2	8	18	32				2	8			

The graph is marked (b) in Fig. 49. It should be noticed that (b) (*i.e.* the curve for $k=2$) is just twice the height of (a) (*i.e.* twice the height of the curve for $k = 1$) at each point.

(c) If $y = \frac{1}{2}x^2$, the curve is just half as high at each point as is (a). [Let student make a table, and actually draw the graph.]

(d) [Let the student draw other curves to represent $y = 3x^2$, $y = 4x^2$, $y = 5x^2$, $y = 10x^2$, $y = \frac{1}{3}x^2$, $y = \frac{1}{10}x^2$, $y = -x^2$, $y = -2x^2$, $y = -\frac{1}{2}x^2$.] Different colors of ink or pencil may be used to advantage for the different curves.

All these should be drawn *on the same sheet of paper* and marked so that they can be recognized.

Ex. 2. $y = kx^3$.

This equation is readily plotted as above. Only the figure is here given; and that only for $k = 1$, $k = 2$, $k = \frac{1}{2}$.

[Let the student make a table for each of these.]

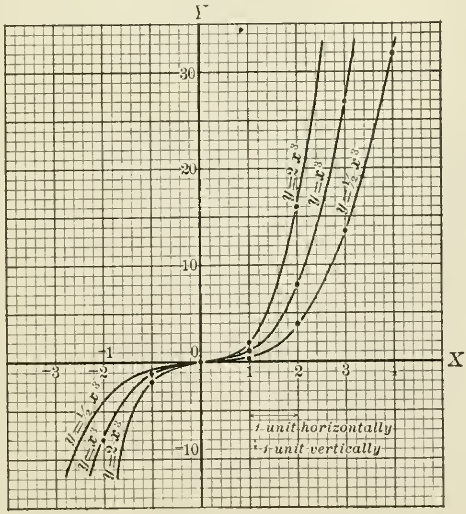


FIG. 50.

Ex. 3. $x^2+y^2=k^2$.

(*a*) If $k=1$, $x^2+y^2=1$.

Take a point P and mark the values of x and y as in the figures:

$$x = OM,\ y = MP.$$

Now $\overline{OM}^2+\overline{MP}^2=\overline{OP}^2$ since OMP is a right-angled triangle. [The student should know that the square of the diagonal side of a right-angled triangle is equal to the sum of the squares of the other two sides. See Tables.]

Hence, $x^2+y^2=OP^2$; but $x^2+y^2=1$; hence, $OP^2=1$ or $OP=1$. Consequently the point P is at a distance 1 from O; the points for which this is true lie on a circle of radius 1 whose center is at O.

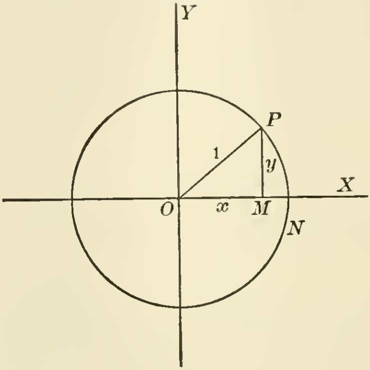


FIG. 51.

The equation $x^2+y^2=1$ is represented by a circle of unit radius about the origin as center.

(*b*) If $k=2$, the reasoning is the same except that $OP=2$; hence, *$x^2+y^2=4$ is represented by a circle of radius 2 about the origin.*

(*c*) *In any case $x^2+y^2=k^2$ is represented by a circle of radius k about the origin*, for $OP=k$, whatever k may be.

EXERCISES III: CHAPTER IX

Plot the following curves:

1. $y=\dfrac{100}{x^2}$.

2. $y=x+\dfrac{20}{x}$.

3. $y=x^2-7x+10$.

4. $y=\dfrac{50}{10-x}$.

5. $y = \frac{50}{x} + \frac{50}{10 - x}$.

6. $y = x + \sqrt{x}$.

7. $y = x^2 + x$.

8. $y = 2x^2 - 9x + 8$.

For various values of k draw the curves:

9. $y = \frac{k}{x^2}$.

10. $y = x^2 + 2x + k$.

11. $(x - k)^2 + y^2 = 1$.

12. $x^2 + (y - k)^2 = 1$.

For various values of k and l, draw the curves:

13. $y = kx + \frac{l}{x}$.

14. $y = x^2 + kx + l$.

15. $(x - k)^2 + (y - l)^2 = 1$.

16. $y = kx - l\sqrt{x}$.

17. Represent graphically the relation between the ratios of each perpendicular side of a right triangle to the hypotenuse.

NOTE. These ratios are called the "sine" and the "cosine" of one of the angles of the triangle; it is suggested that they be denoted here by the letters s and c; and that the base, altitude, and hypotenuse be denoted by b, a, and h, respectively. Show that $s = \frac{a}{h}$, $c = \frac{b}{h}$; then since $a^2 + b^2 = h^2$, show that $s^2 + c^2 = 1$, by dividing both sides by h^2.

18. Represent by a figure the relation between the ratio of the hypotenuse of a right triangle to a perpendicular side, and the ratio of the other perpendicular side to the former.

NOTE. Denote these ratios by $x\left(=\frac{h}{b}\right)$ and by $t\left(=\frac{a}{b}\right)$ (called "tangent"). Prove first that $x^2 = 1 + t^2$.

19. Represent by a figure the relation between the ratio (c) of one side of a right triangle to the hypotenuse and the ratio (x) of the hypotenuse to that same side. Show first that $x \times c = 1$.

20. Find the maximum rectangle of given perimeter k. (Solve for various special values of k; then note the results, and state generally.)

SUMMARY OF CHAPTER IX: VARIATION; INDETERMINATE EQUATIONS, pp. 237–251

Simple Variation: equivalence to proportion between variables; also to equation $y = kx$; figure, straight line. § 122, pp. 237–238.

Linear Variation: equivalence to equation $y = ax + b$; figure, straight line.

General Linear Equation: figure always straight line. § 123, pp. 238–239.

Inverse Variation: constancy of product of two variables; typical problem; Fig. 43, inverse variation curve. Exercises I. § 124, pp. 239–242.

Indeterminate Equations: definition; typical problems; figures. Exercises II (for graphical solution). § 125, pp. 242–247.

General Indeterminate Equations: general case of variation; plotting curves by assignment of values to one letter.

Special Types: type $y = kx^2$, *i.e.* variation as the square; $y = kx^3$; circle $x^2 + y^2 = r^2$, center at origin, radius r. Exercises III. § 126, pp. 247–251.

CHAPTER X. SIMULTANEOUS EQUATIONS INVOLVING QUADRATICS

PART I. ONE LINEAR AND ONE QUADRATIC

127. Introduction. When two simultaneous equations are given, one of which is linear, the other quadratic, it is usually best to use the method of substitution, similar to that of § 90, p. 170.

Ex. 1. A wagon bed is to be made to hold 2 cubic yards. It must be 4 feet wide and six times as long as it is high. Find the dimensions of the wagon bed.

Let x be the height of the bed in feet and y the length in feet. Since the length is to be six times the height, we have

(1) $$y = 6x.$$

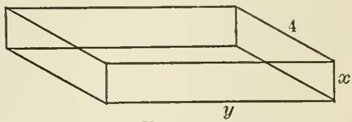


FIG. 52.

The *volume* is $4xy$ (in cubic feet), for the volume of a rectangular box is the product of its three dimensions. The volume is to be 2 cu. yd. = 54 cu. ft.

(2) $$\therefore\ 4xy = 54, \quad \text{or} \quad 2xy = 27.$$

(This is called a *quadratic* equation *in x and y*, for the *sum* of the exponents of x and y is 2.)

Let us now solve the equations we have found:

$$\begin{cases} y = 6x, & (1) \\ 2xy = 27. & (2) \end{cases}$$

Substitute $6x$ for y from (1) in (2):

$$2x(6x) = 27,$$

or, $$12x^2 = 27, \quad \text{whence,} \quad x^2 = \tfrac{9}{4}, \quad \text{or,} \quad x = \pm\tfrac{3}{2}.$$

If $x = +\frac{3}{2}$,

$y = 6x = 9$.

Hence, the result is $x = \frac{3}{2}$, $y = 9$.

Check:

$y = 6x$; $9 = 6 \times \frac{3}{2}$ (correct).

$2xy = 27$; $2 \cdot \frac{3}{2} \cdot 9 = 27$ (correct).

If $x = -\frac{3}{2}$,

$y = 6x = -9$.

Hence, the result is $x = -\frac{3}{2}$, $y = -9$.

Check:

$y = 6x$; $-9 = 6(-\frac{3}{2})$ (correct).

$2xy = 27$;

$2(-\frac{3}{2})(-9) = 27$ (correct).

In this example only the first set of answers has a real meaning. In other applications of these equations both sets of answers may be useful.

Check: The dimensions of the wagon bed are: width = 4 ft; height = $1\frac{1}{2}$ ft; length = 9 ft. The volume is then $4 \times 1\frac{1}{2} \times 9$ cu. ft. = 54 cu. ft. = 2 cu. yd., as required.

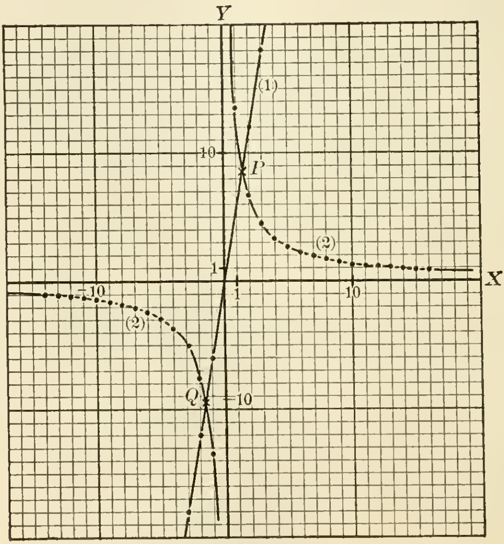


Fig. 53.

We may also draw a figure, Fig. 53, as on p. 239. Thus, (1) ($y = 6x$) is a straight line, as drawn. (Compare pp. 141, 160.) To draw (2) we give x various values and find the corresponding values of y. (See p. 239 and § 126, p. 247)

x	1	2	3	4	5	6	7	8	9	10	&c.	-1	-2	-3	-4	&c.
y	$\frac{27}{2}$	$\frac{27}{4}$	$\frac{27}{6}$	$\frac{27}{8}$	$\frac{27}{10}$	$\frac{27}{12}$						$-\frac{27}{2}$	$-\frac{27}{4}$			
y (reduced)	13.5	6.75	4.5	3.37	2.7	2.41						-13.5	-6.75			

[Let the student fill in the blank spaces and continue the table.]

The picture for (2) drawn from this table, as on p. 254, is as shown; it has two parts and is surely not straight. In fact, comparing with § 124, p. 239, we see that (2) is a *curve of inverse variation.*

Every point on the straight line (1) *gives a pair of numbers that satisfy* $y = 6x$.

Every point on the inverse variation curve (2) *gives a pair of numbers that satisfy* $2xy = 27$.

The points that lie on both (1) *and* (2) *satisfy both equations,* i.e. *each* **point of intersection** *gives a pair of numbers that are a* **pair of simultaneous solutions** *of* (1) *and* (2). Compare § 86, pp. 159–160.

These points in the figure are P and Q.

P gives ($x = 1\frac{1}{2}$, $y = 9$) Q gives ($x = -1\frac{1}{2}$, $y = -9$), which are in fact the answers found above.

This serves as a check on the work. We shall later find this graphical process invaluable in solving approximately, in difficult examples. Compare p. 276.

128. To find the **degree of a term**, add together the exponents of each of the important (*i.e.* unknown) letters. (See § 83, p. 152.)

The **degree of an equation** is the degree of its term of highest degree after it is freed of fractions and radical signs, and is simplified as far as possible (See p. 152.)

An equation of the **first** degree in the important letters is called a **linear** or **simple** equation. (See p. 152.)

An equation of the **second** degree in the important letters is called a **quadratic equation**. (See p. 152 and compare p. 208.)

In Part I of this chapter we consider pairs of equations, one of which is *linear*, the other *quadratic*.

129. Rule. *To solve for two unknown letters in a pair of equations, one of which is linear, the other quadratic:*

(1) *Solve the linear equations for y (or x) in terms of x (or y).*

(2) *Substitute in the quadratic equation for y (or x) the value just found.*

(3) *The new equation will be a quadratic in x (or y), or else a linear equation (in rare cases).*

(4) *Solve this equation for x (or y), and substitute both values found in the linear equation to find the values of y (or x).*

(5) *Draw the figure as a check on the work.*

(6) *Substitute each pair of answers in the original equations as a complete check.*

Notice that there will usually be *two* pairs of solutions, for we solve a quadratic equation during the process. Care should be taken to *pair off the values of x and the values of y that belong together.* In doing this the figure will be of help in avoiding errors.

Ex. 1.
$$\begin{cases} 2x + y = 5, & (1) \\ x^2 + y^2 = 25. & (2) \end{cases}$$

Solve (1) for y: $y = 5 - 2x.$

Substitute $(5 - 2x)$ for y in (2):

$$x^2 + (5 - 2x)^2 = 25,$$

or, $$x^2 + 25 - 20x + 4x^2 = 25,$$

whence, $5x^2 - 20x = 0$, or $5x(x - 4) = 0$.

Hence, $5x = 0$, or $x - 4 = 0$, that is, $x = 0$, or $x = 4$.

If $x = 0$, If $x = 4$,

$y = 5$ [from (1)]. $8 + y = 5$, or $y = -3$ [from (1)].

The pairs of solutions are therefore $(x = 0,\ y = 5)$ and $(x = 4,\ y = -3)$.

Check:

$$\begin{Bmatrix} x = 0 \\ y = 5 \end{Bmatrix} \text{ in } \begin{Bmatrix} 2x + y = 5 \\ x^2 + y^2 = 25 \end{Bmatrix} \text{ give } \begin{Bmatrix} 2 \cdot 0 + 5 = 5 \\ 0^2 + 5^2 = 25 \end{Bmatrix} \quad \text{(correct).}$$

$$\begin{Bmatrix} x = 4 \\ y = -3 \end{Bmatrix} \text{ in } \begin{Bmatrix} 2x + y = 5 \\ x^2 + y^2 = 25 \end{Bmatrix} \text{ give } \begin{Bmatrix} 2 \cdot 4 + (-3) = 5 \\ 4^2 + (-3)^2 = 25 \end{Bmatrix} \quad \text{(correct).}$$

The graph is as shown in Fig. 54: (1) is a straight line, drawn as in § 80, p. 140, etc.

(2) is a circle with radius 5, and center O, as shown; see p. 250, where the equation $x^2 + y^2 = 25$ is studied. The values ($x = 0$, $y = 5$) and ($x = 4$, $y = -3$) *as we have paired them* correspond to the *points of intersection.* It will be instructive for the student to see what happens if he pairs off the values *incorrectly:* thus, ($x = 0$, $y = -3$) and ($x = 4$, $y = 5$); do these give points on the curves? Do these pairs satisfy the given equations?

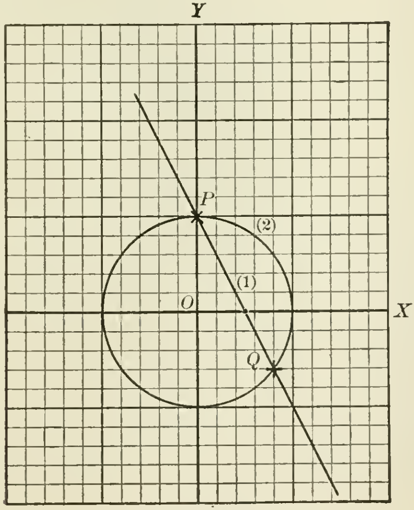


FIG. 54.

Ex. 2.
$$\begin{cases} x + y = 10, & (1) \\ x^2 + y^2 = 25. & (2) \end{cases}$$

Solve (1) for y: $\qquad y = 10 - x.$

Substitute in (2) $\qquad x^2 + (10 - x)^2 = 25,$

or, $\qquad x^2 + 100 - 20x + x^2 = 25,$

or. $\qquad 2x^2 - 20x + 75 = 0.$

The solutions of this equation are *imaginary*, for $a = 2$, $b = -20$, $c = 75$ in the notation of § 114, p. 212.

$$b^2 - 4ac = 400 - 4 \times 2 \times 75 = -200.$$

Hence, there is no value of x among numbers we know at present that satisfies the equation. (See § 113, p. 211; also Appendix, § 31.)

The original example therefore has no solutions. This is clearly brought out by the graph (Fig. 55): (2) is a circle of radius 5 and center O, as before; (1) is a straight line, as shown. It is clear that these two curves *have no point of intersection;* hence, there is no pair of numbers that satisfy both equations. (See p. 211.)

Later (see Appendix, § 37), it is desirable to solve the quadratic. We can then proceed to get answers, but these would be meaningless to the student at present.

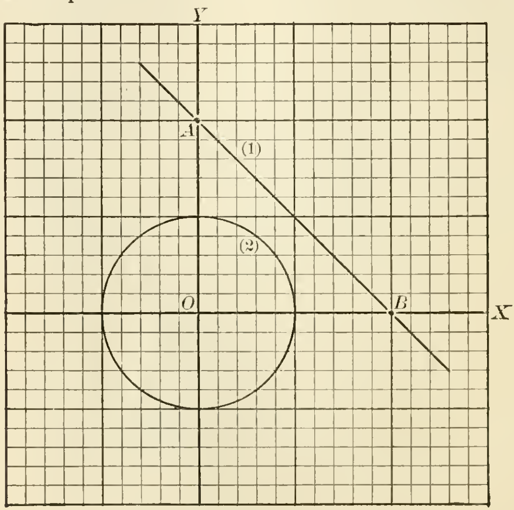


Fig. 55.

Ex. 3. $$\begin{cases} 4x + 3y = 25, & (1) \\ x^2 + y^2 = 25. & (2) \end{cases}$$

Solve for y (1): $$y = \frac{25 - 4x}{3}.$$

Substitute in (2): $$x^2 + \left(\frac{25 - 4x}{3}\right)^2 = 25,$$

or, $$25x^2 - 200x + 625 = 225.$$

Divide by 25: $$x^2 - 8x + 25 = 9,$$

Transpose 9: $$x^2 - 8x + 16 = 0,$$
or, $$(x - 4)^2 = 0.$$

Hence, $x = 4$, there being only one solution. (See § 112, p. 209; this is the case of "equal roots.")

Since $x = 4$, $y = 3$ from (1). The only pair of solutions is therefore $(x = 4, y = 3)$.

Check:

$$\left\{\begin{matrix} x = 4 \\ y = 3 \end{matrix}\right\} \text{ in } \left\{\begin{matrix} 4x + 3y = 25 \\ x^2 + y^2 = 25 \end{matrix}\right\} \text{ gives } \left\{\begin{matrix} 4 \cdot 4 + 3 \cdot 3 = 25 \\ 4^2 + 3^2 = 25 \end{matrix}\right\} \text{ (correct).}$$

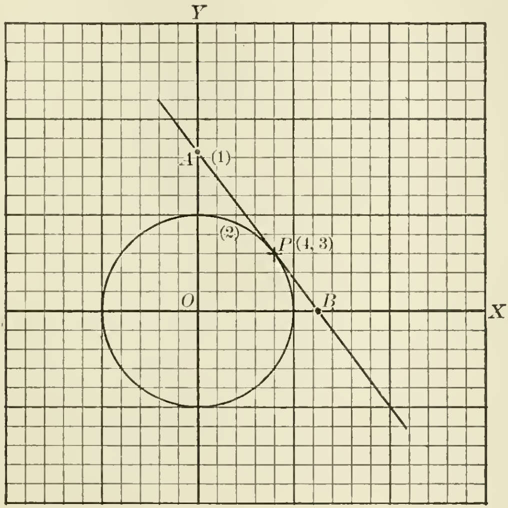


FIG. 56.

Figure 56 makes clear why there is but one answer. The equation (2) gives the same circle as before, *i.e.* radius 5, center O. The straight line (1) goes through the point A: $(x = 0, y = 8\frac{1}{3})$ and B: $(x = 6\frac{1}{4}, y = 0)$. It also goes through $(x = 4, y = 3)$. If carefully drawn as shown here, it is seen that the straight line and the circle have only one point P in common.

The straight line AB is said to be **tangent** to the circle.

In drawing a graph, if the equation is not one we have already studied, it must be plotted out by points as in § 126, pp. 247–248.

EXERCISES I: CHAPTER X

Solve the following pairs of equations when a solution exists. Note in each case whether the quadratic in one variable obtained in the course of the solution has equal or distinct roots. Always draw the corresponding graphs, check the numerical results by substitution in the given equations, and note the interpretation of two, one, or no roots:

* **1.** $\begin{cases} x^2+y^2=34, \\ x+y=2. \end{cases}$

* **2.** $\begin{cases} x^2+y^2=25, \\ x+y+7=0. \end{cases}$

* **3.** $\begin{cases} x^2+y^2=169, \\ 5x+12y=169. \end{cases}$

* **4.** $\begin{cases} x^2+y^2=17, \\ x+y=5. \end{cases}$

† **5.** $\begin{cases} xy=12, \\ x+y=8. \end{cases}$

† **6.** $\begin{cases} xy=9, \\ x+y=6. \end{cases}$

† **7.** $\begin{cases} xy=15, \\ x-y=2. \end{cases}$

‡ **8.** $\begin{cases} 2x+3y=9, \\ x^2-y^2=8. \end{cases}$

† **9.** $\begin{cases} xy=6, \\ 3x-2y=16. \end{cases}$

§ **10.** $\begin{cases} y^2=4x, \\ y=8. \end{cases}$

‡ **11.** $\begin{cases} x^2-y^2=8, \\ 2x+y=7. \end{cases}$

‡ **12.** $\begin{cases} x^2-y^2=12, \\ 2x-y=6. \end{cases}$

13. $\begin{cases} 9x^2+16y^2=25, \\ x+y=2. \end{cases}$

14. $\begin{cases} 9x^2+16y^2=25, \\ x+y=0. \end{cases}$

15. $\begin{cases} x^2+xy=15, \\ 2x+y=8. \end{cases}$

16. $\begin{cases} x^2+xy=21, \\ x-y=-1. \end{cases}$

17. $\begin{cases} xy+y^2=-2, \\ x+2y=1. \end{cases}$

18. $\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}=5, \\ 4x-3y=1. \end{cases}$

19. $\begin{cases} x+y+xy=11, \\ x-y=1. \end{cases}$

20. $\begin{cases} y^2-10x^2=15, \\ y-3x=2. \end{cases}$

* See § 126, p. 248.
† See § 124, p. 240.
‡ See § 131, p. 269.
§ See § 126, p. 248.

130. Ordinary Quadratic. The following special example leads to a new figure for graphical solution of quadratic equations in one letter.

Ex. 1. $\begin{cases} y = 2x + 8, & (1) \\ y = x^2. & (2) \end{cases}$

Substitute for y from (1) in (2):

$$2x + 8 = x^2, \quad \text{or,} \quad x^2 - 2x - 8 = 0.$$

Solving, we get $\qquad x = -2, \quad \text{or,} \quad x = +4.$

If $x = -2$, $\qquad$ If $x = +4$,

$y = 2x + 8 = 2 \cdot (-2) + 8 = 4. \qquad y = 2 \cdot 4 + 8 = 16.$

Check:

$$\begin{Bmatrix} x = -2 \\ y = 4 \end{Bmatrix} \text{ in } \begin{Bmatrix} y = 2x + 8 \\ y = x^2 \end{Bmatrix} \text{ give } \begin{Bmatrix} 4 = 2(-2) + 8 \\ 4 = (-2)^2 \end{Bmatrix} \text{ (correct).}$$

$$\begin{Bmatrix} x = 4 \\ y = 16 \end{Bmatrix} \text{ in } \begin{Bmatrix} y = 2x + 8 \\ y = x^2 \end{Bmatrix} \text{ give } \begin{Bmatrix} 16 = 2 \cdot 4 + 8 \\ 16 = 4^2 \end{Bmatrix} \text{ (correct).}$$

The figure is as shown on p. 264 : (1) is a straight line (draw it); (2) is as shown (see § 95, p. 183).

From the figure there are two points of intersection, P: $(x = -2, y = 4)$ and Q : $(x = 4, y = 16)$; these pairs of numbers are therefore solutions, as we found above.

From this example it is clear that the solution of the quadratic equation

$$x^2 - 2x - 8 = 0$$

gives the same values of x as are given for x by the simultaneous pair

$$\begin{cases} y = x^2, \\ y = 2x + 8. \end{cases}$$

Likewise, the values of x found in $x^2 + px + q = 0$ are also found from

$$\begin{cases} y = x^2, \\ y = -px - q, \end{cases}$$

for, substituting for y, the last pair give the previous equation. The advantage of this graphical picture over

those in Chapter VIII is that *the curve* $y = x^2$ *is the same for all quadratics* by this method. This curve being drawn once for all, nothing remains but to draw the straight line $y = -px - q$.

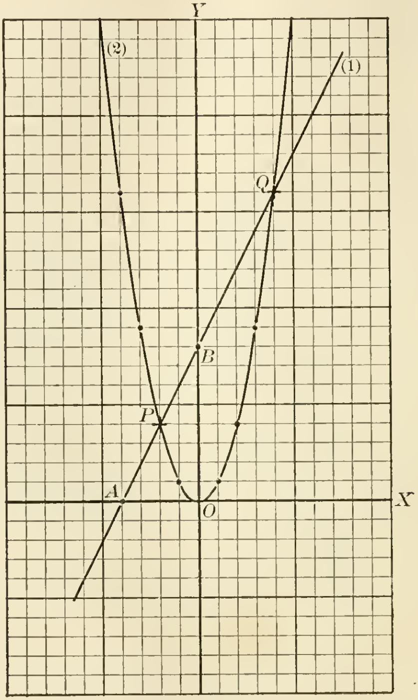


Fig. 57.

Several examples may be drawn in the same figure, as follows:

Solve the following list of examples graphically:

(1) $2x^2 - 9x + 4 = 0$. See p. 203. (4) $x^2 - 4x + 5 = 0$.

(2) $x^2 - 4x + 1 = 0$. See p. 207. (5) $x^2 + 2 = 0$. See p. 211.

(3) $x^2 - 4x + 4 = 0$. See p. 209. (6) $x^2 + 2x + 5 = 0$. See p. 211.

In (1) it is desirable first to reduce the coefficient of x^2 to unity by dividing both sides by 2; this gives

$$(1) \quad x^2 - \frac{9}{2}x + 2 = 0.$$

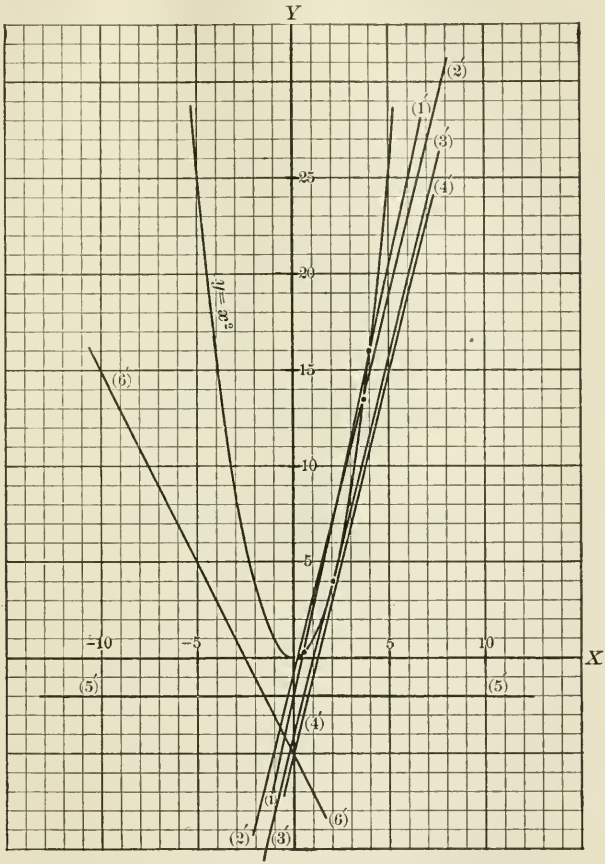


FIG. 58.

As above, (1) corresponds to solving the simultaneous pair

$$\begin{cases} y = x^2, & \text{I} \\ y = \dfrac{9}{2}x - 2 & \text{II} \end{cases}$$

I is the figure drawn above; II is a straight line, marked (1′) in the figure (draw it). The points where these meet are the simultaneous solutions of I and II; *the values of* x at the points of meeting are the solutions of (1), namely $x = \frac{1}{2}$ and $x = 4$ (see p. 203).

The other examples are solved graphically by drawing in the figure the lines

$$(2')\quad y = 4x - 1, \qquad (4')\quad y = 4x - 5,$$

$$(3')\quad y = 4x - 4, \qquad (5')\quad y = -2, \qquad (6')\quad y = -2x - 5.$$

The line (2′) meets the curve at two points *about* $x = \frac{1}{4}$ and $x = 3.7$. This solution is not precisely correct, but is approximately so. (See p. 207.)

The line (3′) meets the curve in only one point; then (3) has only one solution, $x = 2$. (See p. 209.)

The line (4′) does not meet the curve at all; hence (4) has no solution. (Imaginary case; see p. 211.)

The other examples should be compared with their previous solutions.

EXERCISES II: CHAPTER X

Draw the graphs for all the following examples on one diagram; classify them according to the number of answers; estimate the solutions, if there are any, from the figure; check by solving the equations, or if there are no solutions, by computing $b^2 - 4ac$:

1. $x^2 - 9x + 20 = 0.$

2. $x^2 - 3x + 5 = 0.$

3. $x^2 - 3x - 5 = 0.$

4. $x^2 + 8x + 15 = 0.$

5. $x^2 + 8x - 15 = 0.$

6. $4x^2 + 9 - 12x = 0.$

7. $3x^2 - 8x + 5 = 0.$

8. $7x^2 - x + 1 = 0.$

9. $6x^2 + 6x + \frac{3}{2} = 0.$

10. $x^2 - x + 1 = 0.$

11. $x^2 + x - 2 = 0.$

12. $x^2 - 3x + 1 = 0.$

13. Draw a figure for solving the equation $x^2 - 4x + q = 0$ for various values of q from $q = 0$ to $q = 10$ by the method of § 130. Classify the character of the solutions of the equation (§ 115) according to the value of q, estimate solutions, and check.

REVIEW EXERCISES III: CHAPTER X

Solve the following pairs of equations:

1. $\begin{cases} x^2 + xy = 24, \\ x + 2y = 2. \end{cases}$

2. $\begin{cases} p^2 + q^2 = 5, \\ 3p - 5q = 1. \end{cases}$

3. $\begin{cases} m^2 - mn = 3, \\ 2m - n = 4. \end{cases}$

4. $\begin{cases} 2v^2 - 3u^2 = 38, \\ 2v - 3u = 4. \end{cases}$

5. $\begin{cases} k + l + kl = 5, \\ 5k - 2l = 1. \end{cases}$

6. $\begin{cases} \dfrac{x}{y} + \dfrac{y}{x} = \dfrac{13}{6}, \\ x + y = 5. \end{cases}$

7. $\begin{cases} 2zy - z - y = 4, \\ 3z - 10y = 5. \end{cases}$

8. $\begin{cases} x^2 + y^2 + x + y = 54, \\ x - y = 4. \end{cases}$

9. $\begin{cases} n + \dfrac{r}{n} = 8, \\ 3n - r = 6. \end{cases}$

10. The diagonal of a rectangle is 13 inches long. What are its dimensions, if one side is 7 inches longer than the other?

11. It takes 2 hours longer for one pipe to empty a tank than for a second pipe to empty an equal tank. Both pipes together can fill either tank in 1 hour 20 minutes. How long would it take each separately to do so?

12. The fence around the outside of a walk 5 feet wide surrounding a park lot is 340 feet long; the area of the lot itself is 5000 square feet. What are the dimensions of the lot?

13. What integer can be taken such that the sum of all integers from 10 up to and including the chosen one shall be 1230?

Solution. Let n be the chosen number. Then the sum is

$$10 + 11 + 12 + \cdots + (n-1) + n = 1230.$$

Simply reversing this, we get

$$n + (n-1) + (n-2) \cdots + 11 + 10 = 1230.$$

Adding, $(10+n) + (10+n) + \cdots + (10+n) + (10+n) = 2460.$

If there are t terms, we have

$$t\,(10 + n) = 2460.$$

But it is easy to see that

$$10 + t - 1 = n.$$

Solving these two equations, as above, we find,

$$n = 50,\ t = 41.$$

14. If we add together the numbers obtained on starting with $-\frac{5}{2}$ and increasing by unity successively, where must we stop in order to have the sum 8?

15. Solve Ex. 9 of Chapter VIII, Exs. VI, p. 216, by the use of two unknowns.

16. Solve Ex. 18 of Chapter VIII, Exs. VI, p. 217, by the use of two unknowns.

17. Solve Ex. 30 of Chapter VIII, Exs. VI, p. 219, by the use of two unknowns.

18. Solve Ex. 54 of Chapter VIII, Exs. VI, p. 221, by the use of two unknowns.

19. Solve Ex. 51 of Chapter VIII, Exs. X, p. 233, by the use of two unknowns.

20. Solve Ex. 56 of Chapter VIII, Exs. X, p. 233, by the use of two unknowns.

As suggested in Exs. 15–20, many problems solved in previous chapters by means of one unknown may now be solved, in some cases more expeditiously, by the use of two or more unknown numbers; it is recommended that many problems from previous lists be now solved in this manner under the guidance of the teacher.

PART II. SIMULTANEOUS QUADRATICS

131. Simultaneous Quadratics. *Two quadratic equations which are both to hold true are called a pair of simultaneous quadratics.*

We can often solve such pairs by methods similar to those of § 129, p. 256, and §§ 86–89, pp. 159–172. The following examples will illustrate these methods:

Ex. 1. Given
$$\begin{cases} x^2 + y^2 = 16, & (1) \\ y = \frac{1}{6}x^2. & (2) \end{cases}$$

Solve (2) for x^2: $x^2 = 6y$.

Substitute $6y$ for x^2 from (2) in (1):

(3) $y^2 + 6y = 16.$

Complete the square: $y^2 + 6y + 9 = 25,$

or, $y + 3 = \pm 5.$

Hence, either, $y = 2$, or $y = -8$.

If $y = 2$,	If $y = -8$,
$x^2 = 12$,	$x^2 = -48$,
and $x = \pm\sqrt{12}$.	$x = \pm\sqrt{-48}$ (imaginary).

The real solutions are $(x = +\sqrt{12}, y = 2)$ and $(x = -\sqrt{12}, y = 2)$.

The other expressed solutions are meaningless to the student at present. They are $(x = \sqrt{-48}, y = -8)$ and $(x = -\sqrt{-48}, y = -8)$; we shall not regard them as solutions here (see Appendix, §§ 31–38).

Check for real solutions:

$$\begin{Bmatrix} x = \pm\sqrt{12} \\ y = 2 \end{Bmatrix} \text{in} \begin{Bmatrix} x^2 + y^2 = 16 \\ y = \frac{1}{6}x^2 \end{Bmatrix} \text{gives} \begin{Bmatrix} 12 + 4 = 16 \\ 2 = \frac{1}{6} \times 12 \end{Bmatrix} \text{(correct).}$$

Graph. The figure is easy to draw, for each curve was drawn above: (1) is a circle of radius 4 about O as center (see p. 250); (2) is a curve of the kind drawn in § 126, p. 248.

The table for $y = \frac{1}{6}x^2$ is

x . .	0	± 1	± 2	± 3	± 4	± 5	± 6	± 7	± 8	± 9	± 10	etc.	± 15	etc.	± 20
y . .	0	$\frac{1}{6}$	$\frac{2}{3}$	$1\frac{1}{2}$	$2\frac{2}{3}$	$4\frac{1}{6}$	6	$8\frac{1}{6}$	$10\frac{2}{3}$	$13\frac{1}{2}$	$16\frac{2}{3}$	etc.	$37\frac{1}{2}$	etc.	$66\frac{2}{3}$

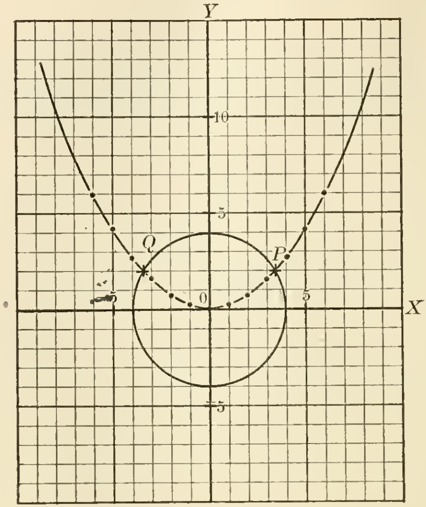


Fig. 59.

The whole figure is as drawn. From it we note the *two* points of intersection, P and Q: P is ($x = 3.5$ (about), $y = 2$) and Q is ($x = -3.5$ (about), $y = 2$).

These agree with the results above as closely as we could expect; in fact $\sqrt{12} = 3.464 \cdots$ as will be found by the ordinary process for square root in arithmetic.

The solution of this example illustrates the method of *substitution*. (Compare p. 170 and p. 256.)

Ex. 2.
$$\begin{cases} x^2 + y^2 = 25, & (1) \\ x^2 - y^2 = 7. & (2) \end{cases}$$

We proceed either as before, by substitution, or as follows:

$$\begin{array}{r|l} 1 & x^2 + y^2 = 25 \\ 1 & x^2 - y^2 = \ \ 7 \\ \hline \end{array} \qquad\qquad \begin{array}{r|l} 1 & x^2 + y^2 = 25 \\ -1 & x^2 - y^2 = \ \ 7 \\ \hline \end{array}$$

$$\begin{aligned} 2x^2 &= 32 \\ x^2 &= 16 \\ x &= \pm 4 \end{aligned} \qquad\qquad \begin{aligned} 2y^2 &= 18 \\ y^2 &= 9 \\ y &= \pm 3 \end{aligned}$$

The possible combinations are (compare example 1, pp. 256–257):

$$A: \begin{cases} x = +4 \\ y = +3 \end{cases}; \quad B: \begin{cases} x = +4 \\ y = -3 \end{cases}; \quad C: \begin{cases} x = -4 \\ y = +3 \end{cases}; \quad D: \begin{cases} x = -4 \\ y = -3 \end{cases}.$$

The figure shows these points clearly: equation (1) is the circle drawn before (radius 5, center O); equation (2) gives the following table:

y	± 0	± 1	± 2	± 3	± 4	± 5	± 6	± 7	± 8	± 9	&c.
x	$\pm\sqrt{7}$	$\pm\sqrt{8}$	$\pm\sqrt{11}$	$\pm\sqrt{16}$	$\pm\sqrt{23}$	$\pm\sqrt{32}$					
x (reduced)	± 2.66	± 2.8	± 3.3	± 4	± 4.8	± 5.7					

and the figure is as shown in [(2) in figure].

The points A, B, C, D of intersection correspond to the pairs of solutions just found.

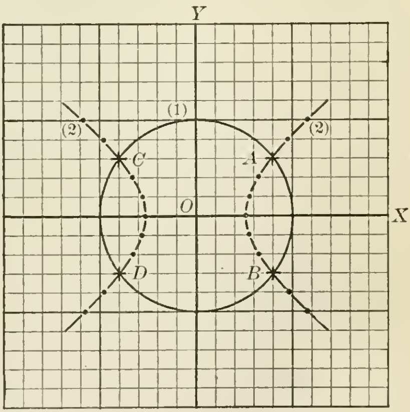


FIG. 60.

Check for

$$\left\{\begin{matrix} x = \pm 4 \\ y = \pm 3 \end{matrix}\right\}:$$

$$\begin{cases} 4^2 + 3^2 = 25, \\ 4^2 - 3^2 = 7. \end{cases}$$

The solution of this illustrates the method of *addition or subtraction*. (Compare p. 163.)

It is *always* successful for pairs of equations of the type

$$ax^2 + by^2 = c.$$

In solving exercises, ingenuity is sometimes required. Often, however, some simple process, usually the method of *substitution* or the method of *addition or subtraction* will suffice in the following exercises. (See also Chapter XII, p. 318, and Appendix, §§ 39–42.)

EXERCISES IV: CHAPTER X

Solve the following equations; always draw the figure:

1. $\begin{cases} y = 5x^2 + 3, \\ x^2 + 2y^2 = 129. \end{cases}$

2. $\begin{cases} a^2 + b^2 = 85, \\ a^2 - b^2 = 77. \end{cases}$

3. $\begin{cases} x^2 + 2xy = 55, \\ 2x^2 - xy = 35. \end{cases}$

[Suggestion. Solve first for x^2 and xy.]

4. $\begin{cases} 3t^2 - 5at = 42, \\ 5t^2 - 4at = 161. \end{cases}$

5. $\begin{cases} u^2 + v^2 = 50, \\ uv = 7. \end{cases}$

[Suggestion. Multiply the second equation by 2, and combine by addition and subtraction with the first equation.]

6. $\begin{cases} m^2 + n^2 = 29, \\ mn = 10. \end{cases}$

7. $\begin{cases} p^2 + q^2 + p + q = 20, \\ pq = -4. \end{cases}$

8. $\begin{cases} 19x^2 - 9y^2 = 603, \\ x^2 + y^2 = 45. \end{cases}$

9. $\begin{cases} x^2 + xy = 56, \\ xy + y^2 = 8. \end{cases}$

10. $\begin{cases} p^2 + 2pq = 21, \\ 2pq + q^2 = 16. \end{cases}$

[Suggestion. Multiply the first equation by 16, the second by 21, and subtract; factor the resulting equation.

Another method is to set $q = pt$: solve both equations for p^2 and compare results.]

11. $\begin{cases} x^2 + xy + y^2 = 19, \\ x^2 - y^2 = 5. \end{cases}$

12. $\begin{cases} x^2 + 2xy + 3y^2 = 6, \\ 2x^2 + 3xy + 4y^2 = 9. \end{cases}$

[Suggestion. Eliminate the terms in xy.]

132. Raising the Graph Vertically. It will often happen that there will be no solution. Examples of the various possibilities follow in § 133.

Let us first notice the effect of writing

(1) $$y = \frac{1}{6}x^2 + k$$

in place of

(2) $$y = \frac{1}{6}x^2$$

in example 1, § 131, where k is some fixed number. If we write

(3) $$y = \frac{1}{6}x^2 + 2,$$

for example, the curve in the figure will be just 2 units vertically above the curve for equation (2), since each value of x makes y just 2 units greater.

The figure is drawn for the equations (2), (3), and also for

(4) $y = \frac{1}{6}x^2 + 5,$ (5) $y = \frac{1}{6}x^2 - 5.$

There is a curious optical *illusion* about such figures, which may deceive the student; although they may not seem to be so, these curves are the *same shape and size*, and any one is formed from any other by simply raising it or lowering it. The student will convince himself of this by cutting out a piece of paper to fit in one of them and then raising or lowering it.

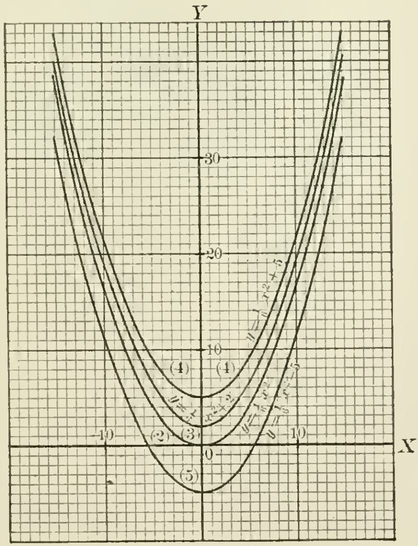


Fig. 61.

133. Various Possibilities Illustrated. Consider now the following examples:

I. $\begin{cases} x^2 + y^2 = 16, & (1) \\ y = \frac{1}{6}x^2. & (2) \end{cases}$ V. $\begin{cases} x^2 + y^2 = 16, & (1) \\ y = \frac{1}{6}x^2 + 10. & (2) \end{cases}$

II. $\begin{cases} x^2 + y^2 = 16, & (1) \\ y = \frac{1}{6}x^2 + 2. & (2) \end{cases}$ VI. $\begin{cases} x^2 + y^2 = 16, & (1) \\ y = \frac{1}{6}x^2 - 4. & (2) \end{cases}$

III. $\begin{cases} x^2 + y^2 = 16, & (1) \\ y = \frac{1}{6}x^2 + 4. & (2) \end{cases}$ VII. $\begin{cases} x^2 + y^2 = 16, & (1) \\ y = \frac{1}{6}x^2 - 5. & (2) \end{cases}$

IV. $\begin{cases} x^2 + y^2 = 16, & (1) \\ y = \frac{1}{6}x^2 + 5. & (2) \end{cases}$ VIII. $\begin{cases} x^2 + y^2 = 16, & (1) \\ y = \frac{1}{6}x^2 - 10. & (2) \end{cases}$

These eight examples are all very similar to example 1, p. 267; and they may be solved in the same way. But a glance at the figure shows that Exs. I and II have two solutions each (A and B, and C and D, in Fig. 62); Ex. III, only one solution (E in Fig. 62); Exs. IV and V, no solutions; Ex. VI, three solutions (F, G, H, in Fig. 62); Exs. VII and VIII, no solutions. For the equation (1) is the same in all of them, while equation (2) differs; in the figure, equation (1) is a circle of radius (4) about O, while equation (2) is the curve of § 131 moved upward or downward. The student will see just when there are solutions by cutting out a piece of paper the shape of the curve and moving it up or down.

Let us solve Ex. VI by the algebraic methods to check these results:

In Ex. VI put $x^2 = 6y + 24$ from (2) in (1); then

$$y^2 + 6y + 24 = 16.$$

Solving this equation, we find $y = -4$ or $y = -2$.

If $y = -4$,	If $y = -2$,
$-4 = \frac{1}{6}x^2 - 4$ from (2),	$-2 = \frac{1}{6}x^2 - 4$,
or $x^2 = 0$, or $x = 0$.	or $x^2 = 12$, or $x = \pm\sqrt{12}$.

The three answers are $\begin{cases} x = 0 \\ y = -4 \end{cases}$; $\begin{cases} x = +\sqrt{12} \\ y = -2 \end{cases}$; $\begin{cases} x = -\sqrt{12} \\ y = -2 \end{cases}$;

which are the points marked F, G, H, respectively, in Fig. 62. Each of the other examples may be solved in a similar manner; the results will be found to agree with the graph.

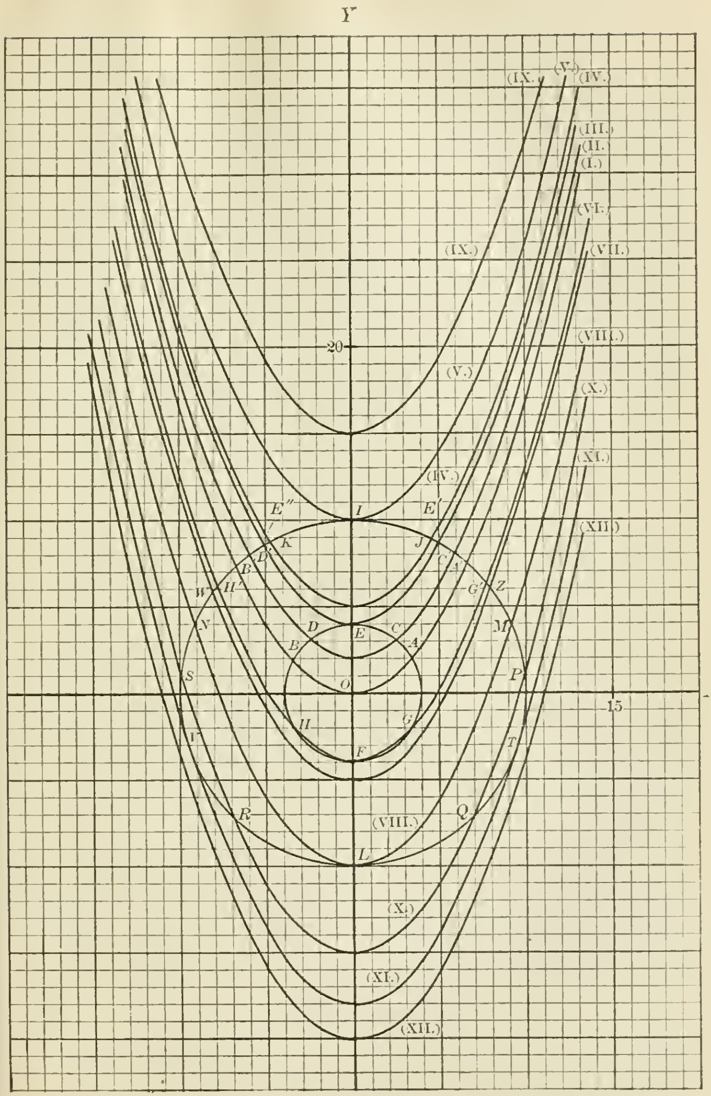

Y
20
15
(IX.)
(V.)
(IV.)
(III.)
(II.)
(I.)
(VI.)
(VII.)
(VIII.)
(X.)
(XI.)
(XII.)
E″
I
E′
K
J
D′
C′
B′
A′
W
H′
G′
Z
N
D
C
M
E
B
A
S
O
P
H
G
F
T
R
Q
L

FIG. 62.

In the same figure we have drawn the new circle

$$x^2 + y^2 = 100,$$

which is a circle of radius 10 about O. This shows some of the details lost because of small size in the other figure. The figure using the larger circle shows graphically the solutions for the following examples:

I′. $\begin{cases} x^2+y^2=100, \\ y=\frac{1}{6}x^2. \end{cases}$

II′. $\begin{cases} x^2+y^2=100, \\ y=\frac{1}{6}x^2+2. \end{cases}$

III′. $\begin{cases} x^2+y^2=100, \\ y=\frac{1}{6}x^2+4. \end{cases}$

IV′. $\begin{cases} x^2+y^2=100, \\ y=\frac{1}{6}x^2+5. \end{cases}$

V′. $\begin{cases} x^2+y^2=100, \\ y=\frac{1}{6}x^2+10. \end{cases}$

VI′. $\begin{cases} x^2+y^2=100, \\ y=\frac{1}{6}x^2-4. \end{cases}$

VII′. $\begin{cases} x^2+y^2=100, \\ y=\frac{1}{6}x^2-5. \end{cases}$

VIII′. $\begin{cases} x^2+y^2=100, \\ y=\frac{1}{6}x^2-10. \end{cases}$

IX′. $\begin{cases} x^2+y^2=100, \\ y=\frac{1}{6}x^2+15. \end{cases}$

X′. $\begin{cases} x^2+y^2=100, \\ y=\frac{1}{6}x^2-15. \end{cases}$

XI′. $\begin{cases} x^2+y^2=100, \\ y=\frac{1}{6}x^2-18\frac{1}{8}. \end{cases}$

XII′. $\begin{cases} x^2+y^2=100, \\ y=\frac{1}{6}x^2+20. \end{cases}$

These are seen by the figure to have the following number of solutions (which can be judged approximately from the figure): I′, two (at A' and B'); II′, two (where in Fig. ?); III′, two (where in Fig. ?); IV′, *two;* V′, *one* (at I); IX′, *none;* VI′, *two* at G' and H'; VII′, *two;* VIII′, *three*, at L, M, N; X′, *four*, at P, Q, R, S; XI′, *two*, at T, V; XII′, *none*.

Of these the most remarkable are X′ with four solutions, and XI′ with its curious two solutions. There are similar ones with the small circle; but they cannot be seen readily in the figure because of the small size. The position in XI′ is determined as follows, so that there shall be just two roots at T and U. We try $y = \frac{1}{6}x^2 + k$ and solve:

$$\begin{matrix} (1) \\ (2) \end{matrix} \quad \begin{cases} x^2 + y^2 = 100, \\ y = \frac{1}{6}x^2 + k. \end{cases}$$

From (2), $\qquad x^2 = 6y - 6k.$

Substitute in (1): $\qquad y^2 + 6y - 6k = 100,$

or, $\qquad y^2 + 6y + (-100 - 6k) = 0.$

The condition that this equation should have equal roots is that $b^2 - 4ac = 0$; we have $a = 1$, $b = 6$, $c = -100 - 6k$; hence,

$b^2 - 4ac = 36 - 4(-100 - 6k) = 0$, or $k = -18\frac{1}{6}$,

which was the value taken in XI′.

It is to be noticed that the value of y may be real although x is imaginary, and *vice versa*, and as in the first case solved (p. 267).

The student will find other possibilities by cutting out the figures shown above and moving them around across each other. Some of these possible positions correspond to fairly complicated pairs of equations.

Other curves may be tried, as suggested in the exercises below.

A complete study of the possibilities — at least a full understanding of them — cannot be hoped for here. It is only after a study of Analytic Geometry, in which such questions are discussed at length, that the student will really appreciate all that is involved.

A few *special rules* are sometimes given; these are really not particularly useful. See Appendix, §§ 39–42.

EXERCISES V: CHAPTER X

Draw figures for Exs. 1–5, note the number of solutions, estimate their numerical values, and solve algebraically:

1. $\begin{cases} x^2 - y^2 = 16, \\ y^2 = 2x - 16. \end{cases}$ (No real solution.)

2. $\begin{cases} x^2 - y^2 = 16, \\ y^2 = 2x - 8. \end{cases}$ (One set of solutions.)

3. $\begin{cases} x^2 - y^2 = 16, \\ y^2 = 2x. \end{cases}$ **4.** $\begin{cases} x^2 - y^2 = 16, \\ y^2 = 2x + 8. \end{cases}$ **5.** $\begin{cases} x^2 - y^2 = 16, \\ y^2 = 2x + 16. \end{cases}$

(How many sets of solutions in each case?)

Similarly the following exercises (6–8):

6. $\begin{cases} x^2 - y^2 = 16, \\ x^2 + y^2 = 9. \end{cases}$ **7.** $\begin{cases} x^2 - y^2 = 16, \\ x^2 + y^2 = 16. \end{cases}$ **8.** $\begin{cases} x^2 - y^2 = 16, \\ x^2 + y^2 = 25. \end{cases}$

Similarly the following exercises (9–16):

9. $\begin{cases} x^2 - y^2 = 1, \\ x^2 + y^2 = 49. \end{cases}$ **10.** $\begin{cases} x^2 - y^2 = 1, \\ (x - 5)^2 + y^2 = 49 \end{cases}$

11. $\begin{cases} x^2-y^2=1, \\ (x-6)^2+y^2=49. \end{cases}$

12. $\begin{cases} x^2-y^2=1, \\ (x-7)^2+y^2=49. \end{cases}$

13. $\begin{cases} x^2-y^2=1, \\ (x-8)^2+y^2=49. \end{cases}$

14. $\begin{cases} x^2-y^2=1, \\ (x-10)^2+y^2=49. \end{cases}$

15. $\begin{cases} x^2-y^2=1, \\ (x-15)^2+y^2=49. \end{cases}$

16. $\begin{cases} x^2-y^2=1, \\ (x-20)^2+y^2=49. \end{cases}$

Treat the following similarly, except that an algebraic solution need not be obtained; instead, check the estimated results by substituting in the equations:

17. $\begin{cases} xy=6, \\ y=3x^2. \end{cases}$ **18.** $\begin{cases} xy=6, \\ y=3x^2-9. \end{cases}$ **19.** $\begin{cases} xy=6, \\ y=3x^2-18. \end{cases}$

134. Graphical Solution. Graphical methods of solution have been mentioned above in practically all cases as a convenient check and as a method for finding approximate answers. There are many problems so difficult that *no method except the graphical one* is really convenient, or even *possible*, with the student's present knowledge. Such is the example below.

Ex. 1.
$$\begin{cases} x^2+y=7, & (1) \\ x+y^2=11. & (2) \end{cases}$$

This pair of equations defies all the methods of *elementary* algebra except the graphical method.

Equation (1) may be written in the form

$$y=-x^2+7. \qquad (1)$$

A table of values of x and y is:

x	± 0	± 1	± 2	± 3	± 4	± 5	± 6	± 7	± 8	± 9	± 10	etc.
y	7	6	3	-2	-9	-18						

and the figure is as shown in Fig. 63. It is the same as the curve of Fig. 31, p. 183, turned upside down and then raised vertically 7 points. Equation (2) may be written

$$x = -y^2 + 11.$$

A table of values of x and y is:

x	11	10	7	2	-5	-14				
y	± 0	± 1	± 2	± 3	± 4	± 5	± 6	± 7	± 8	etc.

and the figure is as shown. It is the same as the curve of Fig. 31, p. 183, turned on its side (*i.e.* turned through 90°) and then moved horizontally 11 points to the right.

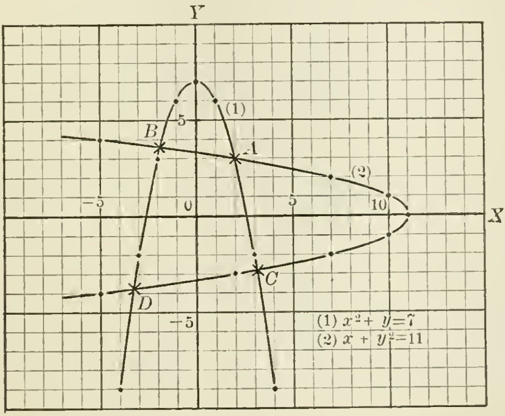


FIG. 63.

These curves are seen by the figure to intersect in *four* points, which, if the figure is carefully drawn, are seen to be about

$$A: \begin{cases} x = 2 \\ y = 3 \end{cases}; \quad B: \begin{cases} x = -1.8 \\ y = 3.6 \end{cases}; \quad C: \begin{cases} x = 3.1 \\ y = -2.8 \end{cases}; \quad D: \begin{cases} x = -3.3 \\ y = -3.8 \end{cases}.$$

The results for B, C, D are not, of course, precisely accurate; the answers can be found to any number of decimal places, however, by drawing the figure on a sufficiently large scale on a large sheet of paper. The exact results to four places are:

$$A: \begin{cases} x=2 \\ y=3 \end{cases}; \quad B: \begin{cases} x=-1.8481 \\ y=+3.5844 \end{cases}; \quad C: \begin{cases} x=3.1313 \\ y=-2.8051 \end{cases}; \quad D: \begin{cases} x=-3.2832 \\ y=-3.7794 \end{cases}.$$

Ex. 2.
$$\begin{cases} x^2+y^2=25, & (1)\\ y=x^3. & (2)\end{cases}$$

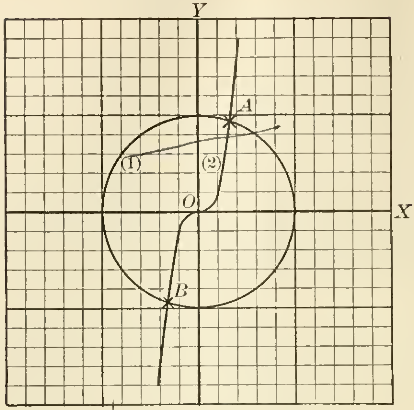


FIG. 64.

The figure is as shown (Fig. 64); (1) is a circle of radius 5, (2) is the cubic curve drawn on p. 189. A table of values for (2) is:

x	0	1	2	3	4	etc.	-3	5	7	etc.	——	-1	-2	etc.
y	0	1	8	27	64		-27	125	343		——	-1	-8	

The solutions are seen to be about $(x=1.7,\ y=4.7)$ and $(x=-1.7,\ y=-4.7)$. They can be found more accurately from a larger figure.

Ex. 3.
$$\begin{cases} y=x^3, & (1)\\ y=5x-2. & (2)\end{cases}$$

Equation (1) is the cubic curve drawn above.

Equation (2) is a straight line through $(x=0,\ y=-2)$ and $(x=5,\ y=23)$.

The figure is as shown (Fig. 65), with $2\frac{1}{2}$ small spaces as unit vertically, and one large space (5 small spaces) as unit horizontally; this is convenient, as will be seen by trying to draw the figure. The solutions are (approximately):

$$A:\begin{Bmatrix} x=2\\ y=8\end{Bmatrix};\quad B:\begin{Bmatrix} x=.4\\ y=.1\end{Bmatrix};\quad C:\begin{Bmatrix} x=-2.4\\ y=-14\end{Bmatrix}.$$

It is interesting to notice that we have really solved the equation

$$(3)\ x^3 - 5x + 2 = 0.$$

For, if we subtract the sides of (2) from those of (1) respectively, we get equation (3). Hence, the values of x just found are the solutions of (3). (Compare p. 261.)

Another method of solving (3) will also be given. We note by trial that $x = 2$ is a solution. *Hence, $x - 2$ is a factor of the left side* (see § 117, p. 223, and Appendix, § 5). Actually dividing $x^3 - 5x + 2$ by $x - 2$ we find,

$$x^3 - 5x + 2 = (x - 2)(x^2 + 2x - 1),$$

hence, (3) is the same as

$$(x - 2)(x^2 + 2x - 1) = 0,$$

whence, either

$$x - 2 = 0, \text{ or } x^2 + 2x - 1 = 0.$$

That is, either

$$x = 2, \text{ or } x = -1 \pm \sqrt{2}.$$

Since $\sqrt{2} = 1.4142 \cdots$,

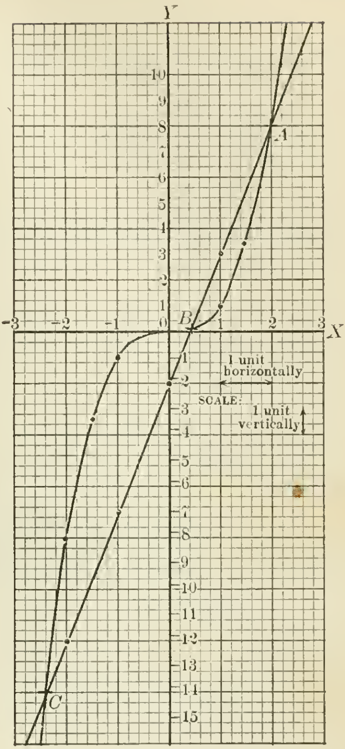


FIG. 65.

the answers are

$$x = 2, \text{ or } x = +.4142 \cdots, \text{ or } x = -2.4142 \cdots,$$

which are more accurate values than those found by our figure.

Subtracting these values in (2), we find

$$y = 8, \text{ or } y = +0.0710 \cdots, \text{ or } y = -14.0710 \cdots;$$

hence, the more precise answers for the simultaneous equations (1) and (2) are:

$$A: \begin{Bmatrix} x = 2 \\ y = 8 \end{Bmatrix}; \quad B: \begin{Bmatrix} x = .4142 \\ y = .0710 \end{Bmatrix}; \quad C: \begin{Bmatrix} x = -2.4142 \\ y = -14.0710 \end{Bmatrix}.$$

EXERCISES VI: CHAPTER X

Solve graphically:

1. $\begin{cases} x^2+3y=7, \\ y^2+3x=1. \end{cases}$

2. $\begin{cases} ab=21, \\ b=a^2-2. \end{cases}$

3. $\begin{cases} y+z=2, \\ z^2-y^2=16. \end{cases}$

4. $\begin{cases} x^2+y^2=25, \\ xy=12. \end{cases}$

5. $\begin{cases} p^2+pq=30, \\ p=5q^2. \end{cases}$

6. $\begin{cases} x^2+y^2=100, \\ y=x^3. \end{cases}$

7. $\begin{cases} mn=10, \\ n=m^3-3. \end{cases}$

8. $\begin{cases} z=t^3, \\ z=t+6. \end{cases}$

9. $x^3-7x-5=0.$

10. $y^3-3y+2=0.$

11. $x^3-x^2-2=0.$

REVIEW EXERCISES VII: CHAPTER X

Solve the following, graphically and analytically:

1. $\begin{cases} z=k^2+7, \\ z^2-25k^2=31. \end{cases}$

2. $\begin{cases} x+y+xy=13, \\ x^2+y^2+xy=43. \end{cases}$

3. $\begin{cases} l^2+3lm=10, \\ lm-m^2=1. \end{cases}$

4. $\begin{cases} u^2+uv+v^2=7, \\ 2v^2-3u^2=5. \end{cases}$

5. $\begin{cases} r^2+s^2=45, \\ \frac{1}{r}+\frac{1}{s}=\frac{1}{2}. \end{cases}$

6. $\begin{cases} y^2=5x, \\ x^2+7y^2=36. \end{cases}$

7. $\begin{cases} y^2=-5x, \\ x^2+7y^2=36. \end{cases}$

8. $\begin{cases} x+y+x^2=4, \\ x^2+4xy+2y^2=17. \end{cases}$

9. $\begin{cases} x^2-y^2=7, \\ xy=12. \end{cases}$

10. $\begin{cases} mn-n^2=6, \\ m^2-mn=15. \end{cases}$

11. $\begin{cases} a^2-b^2=32, \\ (a-3)^2+b^2=13. \end{cases}$

12. $\begin{cases} a^2+b^2=25, \\ (a-5)^2+(b-3)^2=5. \end{cases}$

13. $\begin{cases} x^2+y^2+x+y=98, \\ x^2-y^2+x-y=14. \end{cases}$

Solve graphically:

14. $\begin{cases} x^2 + y = 54, \\ x + y^2 = 32. \end{cases}$

15. $\begin{cases} x^2 - y^2 = 15, \\ xy = 56. \end{cases}$

16. $x^3 - 39x - 70 = 0.$

17. $x^4 - 9x^2 - 4x + 12 = 0.$

18. $s^3 - s^2 - 4s + 5 = 0.$

19. A 24-foot rope is exactly long enough to surround a right triangle, whose hypotenuse is 10 feet long. How long are the other sides?

20. The diagonal of a rectangle whose area is 60 square inches is 13 inches long. What are the length and breadth of the rectangle?

21. I find that I can walk half a mile farther in an hour than John; also it takes me half an hour less to walk 14 miles; how fast do we each walk?

22. The radius of curvature of a concave mirror is 15 cm. How far from the mirror must an object be placed in order that its image be distant 20 cm. farther? (See p. 220.)

23. A loop of twine 30 inches long is to be stretched over four pegs so as to form a rectangle whose area shall be 44 square inches. What must be the sides of the rectangle?

24. The radius of curvature of a concave mirror is r. Where must an object be placed in order that the center of the mirror shall lie halfway between the object and image?

(In solving graphically, choose any convenient value for r.)

25. Find two numbers whose sum is 27, and whose product is 126. (See also p. 224.)

26. Find two numbers whose difference is 19, and whose product is 216.

27. In order that an object 20 cm. farther from a concave mirror than its center may produce an image 30 cm. from the mirror, what radius of curvature must be chosen, and where must the object be placed?

28. A can fold 3000 advertising circulars in three hours less time than B. The two working together can fold 7500 of them in five hours. How many can each fold in one hour?

If there are n numbers, the first of which is a and the last l, such that the differences between consecutive pairs are all equal to the same number d, then it is known (see Chapter XIII, p. 324, § 157) that

$$l = a + (n - 1)d,$$

and that, if s represent the sum of the n numbers,

$$s = \frac{n}{2}(a + l).$$

29. A row of numbers is written down, beginning with 3, such that each number is obtained from the preceding by adding 4. If the sum of the numbers is 55, find all the numbers.

Here a, d, and s are known; we therefore have one linear and one quadratic equation to determine n and l. When n is known, all the numbers can at once be written down.

30. The sum of all the integers from 100 up to a certain integer is 2800. What is this integer?

31. The sum of all the odd integers from 1 to a certain number is 324. What is the last odd number of the list?

SUMMARY OF CHAPTER X: SIMULTANEOUS EQUATIONS INVOLVING QUADRATICS. pp. 253–282

PART I. ONE LINEAR AND ONE QUADRATIC. pp. 253–266.

Figure for one Linear and one Quadratic: straight line and curve as in Chapter IX.

Answers: pairs of values at points of intersection; approximate answers from figure; check on algebraic solution. § 127, pp. 253–255.

Definitions: degree found by adding exponents of unknowns. § 128, p. 255.

Formal Rule for Algebraic Solution: essentially, solve the linear equation for one letter and substitute this value in the quadratic. Exercises I (§§ 127–129). § 129, pp. 255–260.

Second Graphical Method, Ordinary Quadratic: equivalence of roots of $x^2 + px + q = 0$ to roots of $\begin{cases} y = x^2 \\ y = -px - q \end{cases}$; list of examples in one figure. Exercises II. § 130, pp. 261–265.

Review Exercises for Part I, Chapter X: Exercises III. pp. 265–266.

PART II. SIMULTANEOUS QUADRATICS. pp. 267–282.

Simultaneous Quadratics: Figure — two curves; answers — pairs of values at points of intersection; algebraic solution illustrated — methods of Chapter VI. Exercises IV. § 131, pp. 267–270.

Possibilities: possibility of no answers; effect of raising a curve vertically; all cases shown — no answers up to four sets of answers. Exercises V. §§ 132–133, pp. 270–276.

Necessity of Graphical Solution; failure of algebraic methods; recourse to approximate graphical solution. Exercises VI. § 134, pp. 276–280.

Review Exercises for Part II, Chapter X: Exercises VII. pp. 281-282.

CHAPTER XI. RADICALS; FRACTIONAL AND NEGATIVE EXPONENTS

PART I. OPERATIONS; FRACTIONAL AND NEGATIVE EXPONENTS

135. Essential Rules. We have seen (§ 105) that roots may be operated upon in the fractional exponent notation (§ 102) whenever the roots can be found otherwise.

The student should read again and state the definitions of § 94, p. 181, and § 97, p. 186; and he should review §§ 98, 99, 102, 103, 104, 105, with great care. A few important statements are now repeated in some cases in greater detail.

A **rational fraction** is the quotient formed by dividing one integer by another.

A **rational number** is an integer or a rational fraction. Any number that is not rational is called **irrational.** (§ 97.) All rational and irrational numbers are called **real** numbers.

A **radical** is any expressed root, whether rational or irrational, and any expression that contains a radical is called a **radical expression**. The index of the expressed root is the **degree** of the radical.

Thus, $\sqrt{2}$ or $2^{\frac{1}{2}}$, $\sqrt{4}$ or $4^{\frac{1}{2}}$, $\sqrt{b^2 - 4\,ac}$ or $(b^2 - 4\,ac)^{\frac{1}{2}}$ are radicals of the second degree, or *quadratic* radicals; $\sqrt[3]{5}$ or $5^{\frac{1}{3}}$, $\sqrt[3]{x^2}$ or $(x^2)^{\frac{1}{3}}$, are radicals of the third degree, or *cubic* radicals; $3 - \sqrt{2}$, $2\sqrt[3]{5} - 1$, $x + \sqrt[3]{x^2}$ are radical expressions.

A **surd** is a radical that is irrational. A **surd expression** is one that contains surds; such an expression is said to be **surd.** Any radical expression is either surd or rational.

Thus, of the radicals mentioned above, $\sqrt{2}$ and $\sqrt[3]{5}$ are surds; $\sqrt{4}$ is not a surd; $\sqrt[3]{x^2}$ and $\sqrt{b^2-4ac}$ may or may not be surds according to the values of the letters: if $x = 2$, $\sqrt[3]{x^2} = \sqrt[3]{4}$ (surd) but if $x = 8$, $\sqrt[3]{x^2} = \sqrt[3]{64} = 4$ (not surd); in Chapter VIII, § 114, the radical expression $\dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$ is used extensively; in some examples it is a surd expression, in others it is not. (See § 114, p. 212.)

In this Chapter, we shall not deal with even roots of negative numbers, which are often called **imaginary** numbers. (See § 94 and § 113.) The definitions given above exclude such roots. (See Appendix, § 31.)

We shall frequently use the rules of § 104:

$$\text{I.}\quad (x^m)^n = x^{m \cdot n}.$$

$$\text{II.}\quad x^m \times x^n = x^{m+n}.$$

$$\text{III.}\quad x^n \times y^n = (x \cdot y)^n;\quad \text{III } a.\ \frac{x^n}{y^n} = \left(\frac{x}{y}\right)^n.$$

These rules were proved for simple powers only, *i.e.* for positive integral values of m and n. (See, however, § 105.) Let us now try to follow out the work already done by extending our notation as in § 102; and in doing so let us observe the rules I, II, and III, as well as the rules of p. 35, if possible.

136. Meaning of Fractional Exponents. The student already knows the notation of § 102:

(1) $$x^{\frac{1}{2}} = \sqrt{x};$$

(2) $$x^{\frac{1}{3}} = \sqrt[3]{x}, \text{ etc.};$$

or, in general,

(3) $$x^{\frac{1}{n}} = \sqrt[n]{x}.$$

Consider now $(\sqrt[3]{x})^2$; if Rule I is to hold for fractional values of the exponents,

$$(\sqrt[3]{x})^2 = (x^{\frac{1}{3}})^2 = x^{\frac{1}{3} \times 2} = x^{\frac{2}{3}}.$$

In general, by Rule I,

$$(\sqrt[q]{x})^p = (x^{\frac{1}{q}})^p = x^{\frac{1}{q}\times p} = x^{\frac{p}{q}}.$$

If Rule I is to hold for fractional exponents,

$$x^{\frac{p}{q}} = (\sqrt[q]{x})^p.$$

In this work we shall mean, as before, the *positive* answer in case there are two, unless the negative answer is especially indicated by the sign $-$, as in § 94. But in the case of odd roots, where there is only one answer, we shall mean that one, whether it is positive or negative. Even roots of negative numbers are excluded, as stated in § 135, throughout this Chapter.

137. Multiplication and Division; Radicals of same Degree. Consider the product $\sqrt{2}\cdot\sqrt{3}$; this may be written $\sqrt{2}\cdot\sqrt{3} = 2^{\frac{1}{2}}\cdot 3^{\frac{1}{2}}$ or, by Rule III:

$$\sqrt{2}\cdot\sqrt{3} = 2^{\frac{1}{2}}\cdot 3^{\frac{1}{2}} = (2\times 3)^{\frac{1}{2}} = 6^{\frac{1}{2}} = \sqrt{6}.$$

This seems otherwise reasonable, for $(\sqrt{2})^2(\sqrt{3})^2 = (\sqrt{2}\cdot\sqrt{3})^2 = 6$, hence, $(\sqrt{2}\cdot\sqrt{3})^2 = 6$ or $\sqrt{2}\cdot\sqrt{3} = \sqrt{6}$.

In general, $$\sqrt[n]{a}\sqrt[n]{b} = a^{\frac{1}{n}}\times b^{\frac{1}{n}} = (ab)^{\frac{1}{n}} = \sqrt[n]{ab}.$$

Whence, also, $$\frac{\sqrt[n]{ab}}{\sqrt[n]{b}} = \sqrt[n]{a}, \text{ or } \frac{\sqrt[n]{x}}{\sqrt[n]{y}} = \sqrt[n]{\frac{x}{y}}.$$

Even roots of negative numbers are excluded here, as elsewhere in this Chapter, and the agreement of § 94 is maintained. See § 136.

Rule. *Any root of a product of several factors is equal to the product of the similar roots of each of the factors separately.* This rule enables us to multiply radicals of the same degree; it also enables us to *remove* a factor that is a perfect power. Reversing the rule enables us to divide one radical by another of the same degree.

Ex. 1. $\sqrt[3]{3}\times\sqrt[3]{5} = 3^{\frac{1}{3}}\times 5^{\frac{1}{3}} = (3\cdot 5)^{\frac{1}{3}} = 15^{\frac{1}{3}} = \sqrt[3]{15}.$

Ex. 1 *a*. Similarly, $\dfrac{\sqrt[3]{15}}{\sqrt[3]{5}} = \sqrt[3]{3}.$

Ex. 2.

$$\sqrt{2a^3}\sqrt{3b} = (2a^3)^{\frac{1}{2}}(3b)^{\frac{1}{2}} = (2a^3 \times 3b)^{\frac{1}{2}} = (6a^3b)^{\frac{1}{2}} = \sqrt{6a^3b}.$$

Ex. 2 *a*. Similarly, $\dfrac{\sqrt{6a^3b}}{\sqrt{2a^3}} = \sqrt{3b}.$

Check: If $a = 2$, $b = 3$, $\sqrt{2a^3} = \sqrt{16} = 4$; $\sqrt{3b} = \sqrt{9} = 3$.

$$\sqrt{6a^3b} = \sqrt{6 \cdot 8 \cdot 3} = \sqrt{144} = 12;\ 4 \times 3 = 12 \text{ (correct)}.$$

In checking answers by substituting random numbers for letters, if a little care is used in selecting values of the letters, the radicals can always be made *exact*. Do not set any letter equal to 0 or 1, for all powers of either of these numbers are the same and errors would therefore be concealed. In any case, such a check is not complete; it merely shows that the work is *probably* correct.

Values of the letters which would lead to even roots of negative numbers in any instance, are excluded, as stated in § 135.

Ex. 3. $\sqrt{12} = \sqrt{4 \times 3} = (4 \times 3)^{\frac{1}{2}} = 4^{\frac{1}{2}} \times 3^{\frac{1}{2}} = 2 \times 3^{\frac{1}{2}} = 2\sqrt{3}.$

(See p. 188.)

The factor 4, when removed from beneath the radical sign, gives a factor 2 outside it.

Ex. 4. Find the product of $\sqrt{2a^3b} \times \sqrt{6ab^2}$.

$$\sqrt{2a^3b} \times \sqrt{6ab^2} = (2a^3b)^{\frac{1}{2}} \times (6ab^2)^{\frac{1}{2}} = (12a^4b^3)^{\frac{1}{2}} = (4a^4b^2 \times 3b)^{\frac{1}{2}}$$
$$= (4a^4b^2)^{\frac{1}{2}} \times (3b)^{\frac{1}{2}} = 2a^2b \times (3b)^{\frac{1}{2}}, \text{ or } 2a^2b\sqrt{3b}.$$

Here the factors, 4, a^4, b^2, are each perfect squares; they give the factors 2, a^2, b when taken outside the radical sign.

Check: Let $a = 2$, $b = 3$; then the problem and result become

$$(2 \cdot 8 \cdot 3)^{\frac{1}{2}}(6 \cdot 2 \cdot 9)^{\frac{1}{2}} = 4 \cdot 2 \cdot 3\sqrt{3 \cdot 3}, \text{ or, } (48)^{\frac{1}{2}}(108)^{\frac{1}{2}} = 24 \cdot 3,$$

or $(48 \times 108)^{\frac{1}{2}} = 72$, or $(5184)^{\frac{1}{2}} = 72$ (correct).

The real justification for the work done above consists in the fact that any surd radical can be expressed as nearly as we please by a decimal fraction, and the products of these approximate values of the radicals follow the rules of § 104 because they are *exact* roots. (See § 105.) A further explanation of these underlying facts is given in the Appendix, § 30. Just now we shall be satisfied with the arguments given above, which show that these operations are reasonable.

Ex. 5. One reverse of Ex. 4 is $2\,a^2b\sqrt{3\,b} \div \sqrt{2\,a^3b}$.

$$\frac{2\,a^2b\sqrt{3\,b}}{\sqrt{2\,a^3b}} = \frac{(4\,a^4b^2 \times 3\,b)^{\frac{1}{2}}}{(2\,a^3b)^{\frac{1}{2}}} = \left(\frac{4\,a^4b^2 \times 3\,b}{2\,a^3b}\right)^{\frac{1}{2}} = (6\,ab^2)^{\frac{1}{2}} = b\,(6\,a)^{\frac{1}{2}}.$$

The multiplication of binomials and longer expressions is easily done by the rules of Chapter IV (see § 98, p. 187).

Ex. 6. $(2+\sqrt{3})(3-2\sqrt{2})$.

$$\begin{aligned}(2+\sqrt{3})(3-2\sqrt{2}) &= 2\,(3-2\sqrt{2})+\sqrt{3}\,(3-2\sqrt{2})\\ &= 6-4\sqrt{2}+3\sqrt{3}-2\sqrt{6}.\end{aligned}$$

EXERCISES I: CHAPTER XI

In the following exercises perform first the operations indicated, if any; then reduce to as simple a form as possible, removing all possible factors outside the radical:

1. $\sqrt{50}$.

2. $\sqrt{252}$.

3. $\sqrt{300\,a^2b^3c^4}$.

4. $\sqrt[3]{48\,x^4y^7z^2}$.

5. $\sqrt[5]{64\,a^9b^{10}z^6}$.

6. $\sqrt[4]{64\,a^9b^{10}z^6}$.

7. $\sqrt[3]{10} \cdot \sqrt[3]{4}$.

8. $\sqrt{3\,x^2y} \cdot \sqrt{4\,xy^2}$.

9. $\sqrt[3]{18\,xy^4a^5b} \cdot \sqrt[3]{5\,a^2yz} \cdot \sqrt[3]{6\,az^3}$.

10. $\sqrt{10\,ax^2y} \cdot \sqrt{5\,a^2yz} \cdot \sqrt{6\,az}$.

11. $(\sqrt{3\,ab})^3$.

12. $(\sqrt[3]{3\,ab})^2$.

13. $\sqrt{(z-x)(x-y)} \cdot \sqrt{(x-y)(y-z)} \cdot \sqrt{(y-z)(z-x)}$.

14. $\dfrac{6\,ab^2}{\sqrt[3]{3\,a^2b^4}}$.

15. $\dfrac{27\,xyz}{\sqrt{3\,yz}}$.

16. $\dfrac{15\,mn\sqrt[3]{pr^2}}{\sqrt[3]{5\,m^2n^2pr}}$.

17. $\dfrac{(z-x)\sqrt[3]{(x-y)^2(y-z)^2}}{\sqrt[3]{(y-z)(z-x)(x-y)}}$.

18. $\dfrac{3\,b^2-2\,bc+c^2}{(3\,b+c)\sqrt{b-c}}$.

19. $(1+\sqrt{2})^2$.

20. $(5-3\sqrt{6})^3$.

21. $(7-\sqrt[3]{5})^2$.

22. $(\sqrt{7}-\sqrt{2})^2$.

23. $(3+2\sqrt{5})(2-3\sqrt{3})$.

24. $(3-\sqrt[3]{2})^3$.

25. $(x+y\sqrt{z})(x\sqrt{z}-y)$.

26. $(2+\sqrt[3]{4})(\sqrt[3]{2}-3)$.

27. $(6+\sqrt{5})(1+\sqrt{2})$.

28. $(2-\sqrt{2})^2+(3+\sqrt{2})^2$.

29. $(2+\sqrt{7})(1-3\sqrt{7})(5\sqrt{7}-19)$.

30. $(6-\sqrt{3}+\sqrt{2})^2-(1+2\sqrt{3}-3\sqrt{2})^2$.

138. Addition; Similar Radicals. Radicals can be added together (or subtracted) only when they involve the *same root* of the *same number*.

Thus, $2\sqrt{3}+5\sqrt{3}=7\sqrt{3}$; but $2\sqrt{3}+3\sqrt{5}$ must remain as it is. We can in any case, of course, find approximately the value of the radicals, and add these approximate values. Thus, $\sqrt{3}=1.732$ (approximately) and $\sqrt{5}=2.236$ (approximately); hence, $2\sqrt{3}+3\sqrt{5}$ $=10.172$ (approximately).

Radical monomials whose radical factors are the same roots of the same quantities are called **similar**, otherwise they are **dissimilar**.

By § 137 many forms which appear dissimilar may be made *similar*. They may then be added.

Ex. 1. Add $7\sqrt{3}+\sqrt{12}-(75)^{\frac{1}{2}}$.

$$7\sqrt{3}+\sqrt{12}-(75)^{\frac{1}{2}}=7\sqrt{3}+\sqrt{4\times 3}-\sqrt{25\times 3}$$
$$=7\sqrt{3}+2\sqrt{3}-5\sqrt{3}=4\sqrt{3}.$$

EXERCISES II: CHAPTER XI

Simplify the following expressions:

1. $3\sqrt{98}-\sqrt{50}+2\sqrt{8}$.

2. $\sqrt{300}-3\sqrt{27}+5\sqrt{12}$.

3. $2(54)^{\frac{1}{2}}+4(216)^{\frac{1}{2}}-(150)^{\frac{1}{2}}$.

4. $4\sqrt[3]{54}-\sqrt[3]{250}-\sqrt[3]{16}$.

5. $(256)^{\frac{1}{3}}+2(4)^{\frac{1}{3}}-2(108)^{\frac{1}{3}}$.

6. $7\sqrt{45}-2(20)^{\frac{1}{2}}+\sqrt{2000}$.

7. $\sqrt{a^2b^3c^2} - \sqrt{bx^2y^2} + \sqrt{b}.$ **9.** $\sqrt[3]{54\,x^2} - \sqrt[3]{16\,x^2} + \sqrt[3]{250\,x^2}.$

8. $6\sqrt{8\,k} - \sqrt{32\,k} + (162)^{\frac{1}{2}}k.$ **10.** $\sqrt[3]{x^3y^2} + \sqrt[3]{27\,y^5} - (64\,y^2z^3)^{\frac{1}{3}}.$

11. $\sqrt{(x^2 - 7\,x + 6)(x - 1)} + \sqrt{4\,x - 24}.$

12. $\sqrt{x^3 + 3\,x^2 + 3\,x + 1} - \sqrt{x^3 + x^2}.$

13. $\sqrt{x^3 - 6\,x^2 + 9\,x} - \sqrt{x^3} + \sqrt{9\,x}.$

139. Reduction to Different Degree. Consider the example $\sqrt[4]{25}$. We have

$$\sqrt[4]{25} = \sqrt[4]{5^2} = (5^2)^{\frac{1}{4}},$$

or, by Rule I,
$$\sqrt[4]{25} = (5^2)^{\frac{1}{4}} = 5^{\frac{2}{4}} = 5^{\frac{1}{2}} = \sqrt{5},$$

which is correct if Rule I is to hold.

In general, $(\sqrt[q]{x^p}) = x^{\frac{p}{q}} = x^{\frac{p\cdot n}{q\cdot n}} = \sqrt[qn]{x^{pn}},$

which enables us to reduce radicals to different degrees.

This rule will appear more reasonable if we notice that, on raising each side of $x^{\frac{p}{q}} = x^{\frac{pn}{qn}}$ to the power qn, we find $\left(x^{\frac{p}{q}}\right)^{qn} x = ^{\left(\frac{pn}{qn}\right)^{qn}}$ or $x^{pn} = x^{pn}$ by Rule I; but the last result is surely correct. If we agree to take the *positive* root when there are two possible values, we may return from the last result to the preceding rule, which is then justified.

By this new rule we can reduce radicals of different degrees to radicals of the same degree.

Ex. 1. Multiply $\sqrt{3}$ by $\sqrt[3]{2}$.

$$\sqrt{3} = 3^{\frac{1}{2}} = 3^{\frac{3}{6}}, \text{ and } \sqrt[3]{2} = 2^{\frac{1}{3}} = 2^{\frac{2}{6}};$$

hence,
$$\sqrt{3} \times \sqrt[3]{2} = 3^{\frac{3}{6}} \times 2^{\frac{2}{6}} = (3^3)^{\frac{1}{6}} \times (2^2)^{\frac{1}{6}} = (27 \times 4)^{\frac{1}{6}}$$
$$= (108)^{\frac{1}{6}}, \text{ or } \sqrt[6]{108}.$$

Ex. 2. Multiply $\sqrt[3]{3\,ab^2}$ by $\sqrt{a^3b}$.

$$\sqrt[3]{3\,ab^2} = (3\,ab^2)^{\frac{1}{3}} = (3\,ab^2)^{\frac{2}{6}} = (3^2\,a^2b^4)^{\frac{1}{6}},$$

and $$\sqrt{a^3b} = (a^3b)^{\frac{1}{2}} = (a^3b)^{\frac{3}{6}} = (a^9b^3)^{\frac{1}{6}}.$$

Hence, $$\sqrt[3]{3\,ab^2} \times \sqrt{a^3b} = (9\,a^2b^4)^{\frac{1}{6}} \times (a^9b^3)^{\frac{1}{6}}$$

$$= (9\,a^{11}b^7)^{\frac{1}{6}} = (a^6b^6 \times 9\,a^5b)^{\frac{1}{6}}$$

$$= (a^6b^6)^{\frac{1}{6}} \times (9\,a^5b)^{\frac{1}{6}}$$

$$= ab(9\,a^5b)^{\frac{1}{6}} \text{ or } ab\sqrt[6]{9\,a^5b}.$$

Ex. 3. Simplify $\dfrac{\sqrt[3]{4\,a^4b^2} - \sqrt{12\,ab^3}}{\sqrt{2\,a}}$.

$$\frac{\sqrt[3]{4\,a^4b^2} - \sqrt{12\,ab^3}}{\sqrt{2\,a}} = \frac{\sqrt[3]{4\,a^4b^2}}{\sqrt{2\,a}} - \frac{\sqrt{12\,ab^3}}{\sqrt{2\,a}} = \frac{(4\,a^4b^2)^{\frac{1}{3}}}{(2\,a)^{\frac{1}{2}}} - \frac{(12\,ab^3)^{\frac{1}{2}}}{(2\,a)^{\frac{1}{2}}}$$

$$= \frac{(16\,a^8b^4)^{\frac{1}{6}}}{(8\,a^3)^{\frac{1}{6}}} - (6\,b^3)^{\frac{1}{2}} = (2\,a^5b^4)^{\frac{1}{6}} - (6\,b)^{\frac{1}{2}}(b^2)^{\frac{1}{2}}$$

$$= (2\,a^5b^4)^{\frac{1}{6}} - b(6\,b)^{\frac{1}{2}}, \text{ or } \sqrt[6]{2\,a^5b^4} - b\sqrt[2]{6\,b}.$$

The general rule in such cases is to reduce all radicals to fractional exponents, and then operate as if the exponents were integers, by the rules of § 104, which are repeated in § 135, p. 285.

EXERCISES III: CHAPTER XI

Perform the operations indicated; simplify:

1. $\sqrt{5} \cdot \sqrt[3]{7}$.

2. $\sqrt[2]{6} \cdot \sqrt[4]{18}$.

3. $\sqrt{5\,xy^3} \cdot \sqrt{5\,x^2y}$.

4. $\sqrt{ab^2c} \cdot \sqrt[3]{3\,ab^2} \cdot \sqrt[6]{6\,abc^3}$.

5. $\sqrt{7\,a^2x} \cdot \sqrt[3]{21\,ax^2}$.

6. $\sqrt{(x-y)(z-x)^2} \cdot \sqrt[3]{(z-x)(x-y)^2}$.

7. $\dfrac{\sqrt{a^3b^2c}}{\sqrt[3]{ab^2c}}$.

8. $\dfrac{\sqrt{54\,xy^3z}}{\sqrt[3]{3\,xyz^2}}$.

9. $\dfrac{\sqrt{6\,xt^3}-\sqrt[3]{16\,x^2yt^2}}{\sqrt{2\,tx}}$.

10. $\dfrac{\sqrt{mnt}-\sqrt[5]{mnt}}{\sqrt[10]{mnt}}$.

11. $(\sqrt[3]{2}-\sqrt{3})^2$.

12. $(\sqrt{5}-\sqrt[3]{2})(\sqrt{2}-\sqrt[3]{5})$.

13. $(\sqrt{2}-\sqrt[6]{3})(\sqrt{2}+\sqrt[6]{3})$.

14. $(\sqrt[5]{12}-\sqrt{2})(\sqrt[5]{44}+2+\sqrt[10]{4608})$.

15. $(\sqrt{3}-\sqrt[4]{2})^3$.

140. Rationalization of Parts. It is very inconvenient to have a radical in the *denominator* of a fraction, since in calculating the value of the fraction the denominator is then a long decimal, by which it is necessary to divide to get the value of the fraction.

Other parts of an expression also occasionally require simplification for special purposes.

The process of **rationalization**, so called, is any justifiable process that frees some part of a radical expression of the radicals contained in it. It is understood that any operation is *justifiable* only if it leaves the value of the given expression unaltered.

141. Rationalization; Monomial Denominator. By far the most important is the rationalization of monomial denominators.

Ex. 1. $\dfrac{1}{\sqrt{3}}=\dfrac{1}{\sqrt{3}}\dfrac{\sqrt{3}}{\sqrt{3}}=\dfrac{\sqrt{3}}{\sqrt{3}\sqrt{3}}=\dfrac{\sqrt{3}}{3}=\dfrac{1}{3}\sqrt{3}.$

As in this example, any monomial denominator can be rationalized by multiplying both numerator and denominator a sufficient number of times by the radical contained in the denominator, — a process which does not alter the value of the expression.

Occasionally a simpler multiplier may be selected by inspection, the purpose being to make the radical in the denominator exact. This will be learned by practice.

Ex. 2. $\dfrac{\sqrt{3b}}{\sqrt{5a}} = \dfrac{\sqrt{3b} \times \sqrt{5a}}{\sqrt{5a} \times \sqrt{5a}} = \dfrac{\sqrt{15ab}}{5a}.$

Check: Let $a = 5$, $b = 3$; then $\dfrac{\sqrt{3b}}{\sqrt{5a}} = \dfrac{3}{5}$, and $\dfrac{\sqrt{15ab}}{5a} = \dfrac{15}{25} = \dfrac{3}{5}.$

Ex. 3. $\dfrac{1}{\sqrt[3]{5}} = \dfrac{1}{5^{\frac{1}{3}}} = \dfrac{1 \times 5^{\frac{2}{3}}}{5^{\frac{1}{3}} \times 5^{\frac{2}{3}}} = \dfrac{5^{\frac{2}{3}}}{5^{\frac{3}{3}}} = \dfrac{(5^2)^{\frac{1}{3}}}{5} = \dfrac{\sqrt[3]{25}}{5}$, or $\dfrac{1}{5}\sqrt[3]{25}.$

Check:

$\sqrt[3]{5} = 1.7^+$, and $\sqrt[3]{25} = 2.92^+$; hence, $\dfrac{1}{\sqrt[3]{5}} = 0.58^+$, and $\dfrac{\sqrt[3]{25}}{5} = 0.58^+.$

Ex. 4. $\dfrac{\sqrt[3]{4}}{\sqrt[2]{2}} = \dfrac{4^{\frac{1}{3}}}{2^{\frac{1}{2}}} = \dfrac{4^{\frac{2}{6}} \cdot 2^{\frac{3}{6}}}{2^{\frac{3}{6}} \cdot 2^{\frac{3}{6}}} = \dfrac{16^{\frac{1}{6}} \cdot 8^{\frac{1}{6}}}{2} = \dfrac{(16 \cdot 8)^{\frac{1}{6}}}{2} = \dfrac{(2^4 \cdot 2^3)^{\frac{1}{6}}}{2}$

$$= \frac{(2^7)^{\frac{1}{6}}}{2} = \frac{(2^6 2)^{\frac{1}{6}}}{2} = \frac{2(2)^{\frac{1}{6}}}{2} = 2^{\frac{1}{6}}, \text{ or } \sqrt[6]{2}.$$

Check: $\dfrac{\sqrt[3]{4}}{\sqrt[2]{2}} = \dfrac{1.59^+}{1.41^+} = 1.12^+$, and $\sqrt[6]{2} = \sqrt[3]{\sqrt{2}} = \sqrt[3]{1.41^+} = 1.12^+.$

Ex. 5. $\dfrac{\sqrt{ab}}{\sqrt[3]{ab^2}} = \dfrac{(ab)^{\frac{1}{2}}}{(ab^2)^{\frac{1}{3}}} = \dfrac{(ab)^{\frac{1}{2}}}{a^{\frac{1}{3}}b^{\frac{2}{3}}} = \dfrac{(ab)^{\frac{1}{2}} \times a^{\frac{2}{3}}b^{\frac{1}{3}}}{a^{\frac{1}{3}}b^{\frac{2}{3}} \times a^{\frac{2}{3}}b^{\frac{1}{3}}} = \dfrac{(a^3b^3)^{\frac{1}{6}} \times (a^4b^2)^{\frac{1}{6}}}{a^{\frac{3}{3}}b^{\frac{3}{3}}}$

$$= \frac{(a^7b^5)^{\frac{1}{6}}}{ab} = \frac{a(ab^5)^{\frac{1}{6}}}{ab} = \frac{(ab^5)^{\frac{1}{6}}}{b}, \text{ or } \frac{1}{b}\sqrt[6]{ab^5}.$$

Check: Let $a = b = 2$; $\dfrac{\sqrt{ab}}{\sqrt[3]{ab^2}} = \dfrac{2}{2} = 1$, and $\dfrac{1}{b}\sqrt[6]{ab^5} = \dfrac{1}{2} \cdot 2 = 1.$

In general, $\dfrac{A}{B\sqrt[n]{C}} = \dfrac{A \times C^{\frac{n-1}{n}}}{BC^{\frac{1}{n}} \times C^{\frac{n-1}{n}}} = \dfrac{AC^{\frac{n-1}{n}}}{BC^{\frac{n}{n}}} = \dfrac{A\sqrt[n]{C^{n-1}}}{BC}.$

A radical expression is said to be in its simplest form when there are no radicals in any denominator and when each radical in the numerator contains no factor that might be taken out.

EXERCISES IV: CHAPTER XI

Rationalize the denominators of the following; check the numerical ones by calculating the value to two or three decimal places before and after rationalization:

1. $\dfrac{1}{\sqrt{2}}$.
2. $\dfrac{1}{\sqrt[3]{2}}$.
3. $\dfrac{\sqrt{6\,kz}}{\sqrt{7\,x}}$.
4. $\dfrac{\sqrt[3]{5\,y^2z}}{\sqrt[3]{2\,x^2}}$.
5. $\dfrac{5}{\sqrt[3]{3}}$.
6. $\dfrac{xyz}{\sqrt[3]{x^2y^2z^2}}$.
7. $\dfrac{6\,ab}{\sqrt{18\,a}}$.
8. $\dfrac{x^2-y^2}{\sqrt{2(x-y)}}$.
9. $\dfrac{a^2+2\,ab+b^2}{\sqrt[3]{3(a+b)^2}}$.
10. $\dfrac{xy-xz}{\sqrt{x}\,\sqrt[3]{y-z}}$.
11. $\dfrac{2\,m^2+3\,mn+n^2}{\sqrt{m+n}\,\sqrt[3]{2\,m+n}}$.
12. $\sqrt{\dfrac{x+1}{x-1}}$.
13. $\dfrac{a^3-1}{\sqrt[4]{a^2+a+1}\cdot\sqrt[3]{a^2-2\,a+1}}$.

142. Rationalization of Binomial Quadratic Denominators. When the denominator is not a monomial it may be very difficult to rationalize it. We shall treat only the case of simple *binomial* denominators, involving *square* roots.

Ex. 1. $$\frac{1}{2+\sqrt{3}}=\frac{1\times(2-\sqrt{3})}{(2+\sqrt{3})(2-\sqrt{3})}=\frac{2-\sqrt{3}}{4-3}=2-\sqrt{3}.$$

Check: $\sqrt{3}=1.732^+$;

hence, $$\frac{1}{2+\sqrt{3}}=\frac{1}{3.732^+}=0.28^-,$$

and $2-\sqrt{3}=2-1.732^+=0.28^-$.

These examples result surprisingly; it is very clear that the final result is much simpler than the given form.

The general principle upon which such examples are done is that $(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})=a-b$.

The radical binomials $\sqrt{a}+\sqrt{b}$ and $\sqrt{a}-\sqrt{b}$ are called **conjugate** if a and b are rational. Their product, $a-b$, is rational.

If the denominator is the *difference* of two quantities, one of which is a radical, multiply both numerator and

denominator by the *sum* of the same two quantities; and *vice versa*. In general, the proper multiplier is the conjugate of the given denominator.

Ex. 2. $$\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}=\frac{(\sqrt{a}+\sqrt{b})\times(\sqrt{a}+\sqrt{b})}{(\sqrt{a}-\sqrt{b})\times(\sqrt{a}+\sqrt{b})}=\frac{a+b+2\sqrt{ab}}{a-b}.$$

Check: Let $$a = 9 \text{ and } b = 4;$$

then $$\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}=\frac{3+2}{3-2}=5,$$

and $$\frac{a+b+2\sqrt{ab}}{a-b}=\frac{9+4+2\cdot 6}{9-4}=\frac{25}{5}=5.$$

Ex. 3. $$\frac{x-\sqrt{y}}{x+\sqrt{y}}=\frac{(x-\sqrt{y})\times(x-\sqrt{y})}{(x+\sqrt{y})\times(x-\sqrt{y})}=\frac{x^2+y-2x\sqrt{y}}{x^2-y}.$$

Check: Put $x = 3$, $y = 4$; student carry out.

Take care not to alter the value of the given expression.

EXERCISES V: CHAPTER XI

Rationalize the denominators of the following; check the first seven by calculating numerical values to two decimal places before and after rationalization:

1. $\dfrac{1}{\sqrt{2}-1}$.

2. $\dfrac{1}{3+2\sqrt{2}}$.

3. $\dfrac{1+\sqrt{2}}{3-2\sqrt{2}}$.

4. $\dfrac{2}{\sqrt{3}-1}$.

5. $\dfrac{2+\sqrt{3}}{2-\sqrt{3}}$.

6. $\dfrac{\sqrt{3}+2\sqrt{2}}{\sqrt{3}-\sqrt{2}}$.

7. $\dfrac{\sqrt{2}+\sqrt{3}}{2-\sqrt{3}}$.

8. $\dfrac{2+\sqrt{5}}{2\sqrt{5}-3}$.

9. $\dfrac{1+2\sqrt{5}}{1-\sqrt{2}}$.

10. $\dfrac{1}{\sqrt{x}+\sqrt{x-1}}$.

11. $\dfrac{\sqrt{l}-\sqrt{l-m}}{\sqrt{l}+\sqrt{l-m}}$.

12. $\dfrac{\sqrt{x^2+y^2}+x-y}{x+y-\sqrt{x^2+y^2}}$.

13. $\dfrac{m\sqrt{a}-n\sqrt{b}}{m\sqrt{a}+n\sqrt{b}}$.

14. $\dfrac{xy}{1-\sqrt{1-x^2y^2}}$.

15. $\dfrac{m}{y-\sqrt{y^2-2mx}}$.

143. Exponent Negative or Zero. It is convenient to use negative numbers as exponents in the following manner.

Since, by Rule II, $$x^m \times x^n = x^{m+n},$$

it follows that $$\frac{x^{m+n}}{x^n} = x^m,$$

or, on putting $m + n = r$, $n = s$, $\dfrac{x^r}{x^s} = x^{r-s}$. (See p. 76.)

This rule works perfectly when r is greater than s.

Thus, $x^5 \div x^3 = x^{5-3} = x^2$, (see p. 76);

and $x^{\frac{1}{2}} \div x^{\frac{1}{3}} = x^{\frac{1}{2}-\frac{1}{3}} = x^{\frac{1}{6}}$, which is correct,

for $$\frac{x^{\frac{1}{2}}}{x^{\frac{1}{3}}} = \frac{x^{\frac{1}{2}} \cdot x^{\frac{2}{3}}}{x^{\frac{1}{3}} \cdot x^{\frac{2}{3}}} = \frac{x^{\frac{3}{6}} \cdot x^{\frac{4}{6}}}{x} = \frac{x^{\frac{7}{6}}}{x} = \frac{x \cdot x^{\frac{1}{6}}}{x} = x^{\frac{1}{6}}.$$

Even in the last example the new rule is the easier.

When we come to examples in which $s > r$, we are led to write down **negative exponents** thus, by the rule above:

$$\frac{x^3}{x^5} = x^{3-5} = x^{-2};$$

but $$\frac{x^3}{x^5} = \frac{1}{x^2};$$

hence, we say that x^{-2} *means* $\dfrac{1}{x^2}$, in order that the rule above may remain true.

In general,

$$\frac{x^n}{x^{2n}} = x^{n-2n} = x^{-n},$$

by the rule above; but

$$\frac{x^n}{x^{2n}} = \frac{1}{x^n};$$

hence, we say that x^{-n} *means* $\dfrac{1}{x^n}$.

Any letter with a negative exponent is equal to the reciprocal of the expression formed by changing the sign of the exponent.

Likewise, if $r = s$, we find

$$\frac{x^r}{x^r} = x^{r-r} = x^0;$$

but

$$\frac{x^r}{x^r} = 1;$$

hence, we say that $x^0 = 1$.

Any number with the exponent 0 *is equal to* 1.

In this statement 0^0 must be excepted, for $\frac{0^r}{0^r}$ has no meaning. The expression 0^0 therefore has no meaning.

144. With this understanding we can work certain problems more quickly.

It is to be remembered that $x^1 = x$.

Ex. 1. $\frac{\sqrt[3]{4}}{\sqrt{2}}$ (Ex. 4, p. 293).

$$\frac{\sqrt[3]{4}}{\sqrt{2}} = \frac{4^{\frac{1}{3}}}{2^{\frac{1}{2}}} = 2^{\frac{2}{3}} \times \frac{1}{2^{\frac{1}{2}}} = 2^{\frac{2}{3}} \times 2^{-\frac{1}{2}} = 2^{\frac{2}{3}-\frac{1}{2}} = 2^{\frac{1}{6}}. \text{ (See p. 293.)}$$

Ex. 2. $\frac{4\,a^2b^3c^0}{a^3b^2}$.

$$\frac{4\,a^2b^3c^0}{a^3b^2} = 4\,a^2b^3 \times a^{-3}b^{-2} = 4\,a^{2-3}b^{3-2}, \text{ since } c^0 = 1,$$

$$= 4\,a^{-1}b = \frac{4\,b}{a}.$$

Ex. 3. $\frac{\sqrt{ab}}{\sqrt[3]{ab^2}}$ (see Ex. 5, p. 293).

$$\frac{\sqrt{ab}}{\sqrt[3]{ab^2}} = \frac{a^{\frac{1}{2}}b^{\frac{1}{2}}}{a^{\frac{1}{3}}b^{\frac{2}{3}}} = a^{\frac{1}{2}-\frac{1}{3}}b^{\frac{1}{2}-\frac{2}{3}} = a^{\frac{1}{6}}b^{-\frac{1}{6}} = a^{\frac{1}{6}}b^{\frac{5}{6}-1}$$

$$= a^{\frac{1}{6}}b^{\frac{5}{6}} \times b^{-1} = \frac{a^{\frac{1}{6}}b^{\frac{5}{6}}}{b}, \text{ or } \frac{1}{b}\sqrt[6]{ab^5}. \text{ (See p. 293.)}$$

It is a useful short rule for the student to remember that *any factor may be changed from one term of a fraction to the other by reversing the sign of its exponent.*

EXERCISES VI: CHAPTER XI

Express without negative or zero exponents:

1. $a^2b^{-3}c^0$.

2. $8\,p^{-6}q^{-3}r^2s^0$.

3. $x^2y^{-3}z^{-\frac{2}{5}}$.

4. $\dfrac{13\,x^3y^2z^{-2}}{7\,l^5m^{-3}x^0}$.

5. $\dfrac{ab^{-2}c^{-\frac{5}{6}}d^2}{a^2e^{-\frac{2}{3}}f^{-3}d^2}$.

6. $\dfrac{2^{-2}x^{-\frac{1}{2}}y^{-3}z^{-\frac{2}{3}}}{5^{-1}x^{-\frac{1}{2}}y^{-2}z^{\frac{1}{2}}}$.

Transfer b, y, z, p, and q from numerator to denominator, or from denominator to numerator:

7. $\dfrac{abx^2y}{2\,p^3}$.

8. $\dfrac{mnp^{-3}}{x^2y}$.

9. $\dfrac{5\,pq^2}{yz^{-3}}$.

10. y^{-1}.

11. y.

12. $\dfrac{1}{2\,y^{-1}}$.

13. $\dfrac{x^2y}{b^2z}$.

14. $\dfrac{6\,xy^{-2}z^{-\frac{3}{5}}}{5\,a^{-4}b^{-\frac{2}{3}}c^{\frac{1}{2}}}$.

15. $\dfrac{12\,a^{\frac{1}{2}}b^{\frac{2}{3}}c^{\frac{3}{4}}p^{-1}q^{-2}}{7\,x^2y^{-3}z^{\frac{5}{6}}}$.

Perform the operations indicated:

16. $5\,x^2y^{-3}z^5 \times 3\,x^{-3}yz^{-4}$.

17. $\dfrac{mn^{-1}p^3}{m^{-3}np^4}$.

18. $2\,x^{\frac{1}{3}}y^2z^{-\frac{1}{2}} \times 3\,x^{-\frac{1}{2}}y^{-2}z^2$.

19. $(30\,x^2y^{-4}z^{\frac{2}{3}}) \div (5\,x^{-2}y^{-4}z^{\frac{1}{2}})$.

20. $7\,a^3b^{-\frac{1}{2}}c^{\frac{2}{3}} \times 5\,a^{\frac{1}{2}}b^2c^{-1} \times 2(abc)^{-\frac{3}{2}}$.

21. $(x^{-3}y^2z^{-\frac{2}{5}})^5$.

22. $(a^{-1}b^{-2}cp^{-\frac{1}{4}})^{-\frac{4}{5}}$.

23. $\sqrt[3]{r^2s^{-3}t^{-\frac{3}{7}}}$.

24. $(x^2 - y^{-2})^3$.

25. $(a^{\frac{1}{2}} + a^{-\frac{1}{2}})^2$.

Change from the sign $\sqrt{\ }$ to fractional exponents; then simplify:

26. $\dfrac{\sqrt{x^{-3}y^2\sqrt{z}}}{\sqrt[3]{x^{-2}y^3z^0}}$.

27. $\dfrac{\sqrt{m^{-3}}\,\sqrt[3]{n^{-2}}}{\sqrt{m^{-5}n^{-3}}}$.

28. $\dfrac{r^{\frac{1}{2}}\sqrt[3]{st^{-1}}}{\sqrt[3]{r^{-1}}(s^{-1}t)^{\frac{1}{3}}}$.

29. $\dfrac{\sqrt{(u+v)^{-2}(a+b)^{-3}}}{\sqrt[3]{(u+v)^{-3}(a+b)^{-2}}}$.

REVIEW EXERCISES VII: CHAPTER XI

Simplify:

1. $\sqrt[5]{64\,x^6y^8z^{10}}$.
2. $\sqrt[3]{108\,x^5y^{10}z^2}$.
3. $\sqrt[3]{60}\cdot\sqrt[3]{18}$.
4. $(\sqrt[3]{2\,k^2l})^4$.
5. $\dfrac{9\,pq^3}{\sqrt{6\,pq}}$.
6. $\dfrac{zt^2}{\sqrt{zt^{-1}}}$.
7. $(\sqrt{r}+\sqrt{s})^2$.
8. $(\sqrt{r}+\sqrt{s})^3$.
9. $\sqrt{3\,a}\cdot\sqrt[3]{5\,b}$.
10. $(\sqrt{7}-\sqrt[3]{3})^3$.
11. $\dfrac{15\,abx^2y}{\sqrt[3]{9\,ab^2x^5y^2}}$.
12. $\dfrac{4^{-\frac{1}{2}}a^0b^{-\frac{1}{2}}}{2^{-1}a^{-2}b^{-\frac{3}{2}}}$.
13. $\sqrt{432\,(y-z)^5ab^9z^2}$.
14. $(a-b\sqrt{c})(c-d\sqrt{e})$.
15. $\dfrac{(x-y)^{-2}(y-z)^{-1}}{\sqrt{(x-y)^{-5}(y-z)^{-3}}}$.
16. $\dfrac{xyz-3\,z\sqrt[3]{x^2y^2}}{\sqrt[3]{xyz^2}}$.
17. $\dfrac{\sqrt{mn}\cdot\sqrt[3]{m^2n^2}}{\sqrt[4]{mn^3}\cdot\sqrt[3]{mn^2}}$.
18. $\sqrt{50(a^3+3\,a^2b+3\,ab^2+b^3)}$.
19. $\sqrt{4\,mnp^2}\cdot\sqrt{18mn^3p}\,.\,\sqrt{3\,p}$.
20. $\sqrt[3]{56\,kr^4}\cdot\sqrt[3]{14\,k^2r}\cdot\sqrt[3]{7\,rs}$.
21. $\sqrt{xyz^3}\cdot\sqrt[3]{6\,x^5y^2z}\cdot\sqrt[4]{3\,x^2y}$.
22. $\sqrt[3]{432}+\sqrt[3]{1024}-\sqrt[3]{1458}$.
23. $3\sqrt{288}-\sqrt{450}-\sqrt{578}$.
24. $\sqrt{x^3-3\,x^2-9\,x+27}+3\sqrt{x+3}$.
25. $(\sqrt[3]{m}+\sqrt[3]{n}-\sqrt[3]{m^2n^2})(\sqrt[3]{m}+\sqrt[3]{n}+\sqrt[3]{m^2n^2})$.
26. $2\sqrt{x^3+ax^2-a^2x-a^3}-\sqrt{x^3-3\,a^2+3\,a-1}-\sqrt{4\,x-4\,a}$.

Rationalize the denominators of the following:

27. $\dfrac{xy}{\sqrt[3]{xy^2}}$.
28. $\dfrac{\sqrt{2}}{\sqrt[3]{6}}$.
29. $\dfrac{\sqrt{a}+\sqrt{b}}{\sqrt{ab}}$.
30. $\dfrac{\sqrt{ab}}{\sqrt{a}-\sqrt{b}}$.
31. $\dfrac{7}{3-\sqrt{2}}$.
32. $\dfrac{2-3\sqrt{2}}{3+2\sqrt{2}}$.
33. $\dfrac{m\sqrt{a}+n\sqrt{b}}{n\sqrt{a}-m\sqrt{b}}$.
34. $\dfrac{\sqrt{x+3}-\sqrt{x-3}}{\sqrt{x+3}+\sqrt{x-3}}$.

Express in terms of integral exponents and roots:

35. $\dfrac{l^2m^{-\frac{3}{5}}n^{-\frac{2}{3}}}{p^2qr^{-2\frac{1}{2}}s^0}$.
36. $\left(\dfrac{a^{-1}b^{-\frac{2}{3}}c^5}{x^{-2}y^{-\frac{1}{5}}}\right)^2\left(\dfrac{-3\,m^{-1}x^2}{p^{-2}y^{\frac{1}{2}}}\right)^{-2}\left(\dfrac{2\,x^3y^5z^{-3}}{ab^2c^{-\frac{2}{3}}}\right)^{-\frac{1}{2}}$.

PART II. APPLICATIONS: RADICAL EQUATIONS

145. Radical Equations. Equations involving radical expressions are usually to be solved by clearing of the radicals, which can be done by raising both sides of the equation to the same power. Practice in arranging the work will make it easier to see just how this should be done.

In the first place, it is desirable for the student to see that false answers may arise which might be deceptive.

Ex. 1. $\sqrt{x}+2=0$.

Transpose 2: $$\sqrt{x}=-2.$$

Squaring both sides, we find that the only possible solution is:

$$x=4.$$

However, $x=4$ does *not* satisfy the original equation; for if we put 4 for x, we get

$$\sqrt{4}+2=0,$$

$$2+2=0,$$

which is incorrect. Hence, the given equation *has no solution.* This is upon our previous understanding that $\sqrt{4}$ means $+2$ and not -2 (see p. 181).

In order to make the truth clear, let us set

$$l=\sqrt{x}+2,$$

where l means the left side in example 1, and let us make a table of values of x and l.

x	0	1	2	3	4	5	6	7	8	9	10		-1	$-a$
l	2	3	3.4	3.7	4	4.2				5			imag.	imag.

The figure is then as drawn (Fig. 66). It is clear that the left side (l) is never zero; in fact, every value of l is greater than 2, for we *add* $\sqrt{x}$, which is positive, to 2 to get l; hence, the given equation has no solution.

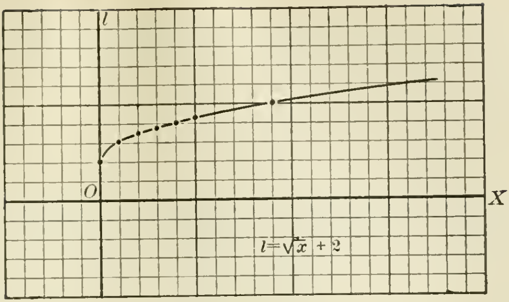


Fig. 66.

At the same time, consider the new expression

$$l' = -\sqrt{x} + 2$$

which results by taking the *negative* square root; the similar table is

x	0	1	2	3	4	5	6	7	8	9			-1	$-a$
l'	2	1	0.6	0.3	0	-0.2				-1			imag.	imag.

Figure 67 shows the values of both l and l': it is clear that these are the upper and lower parts, respectively, of one single curve.

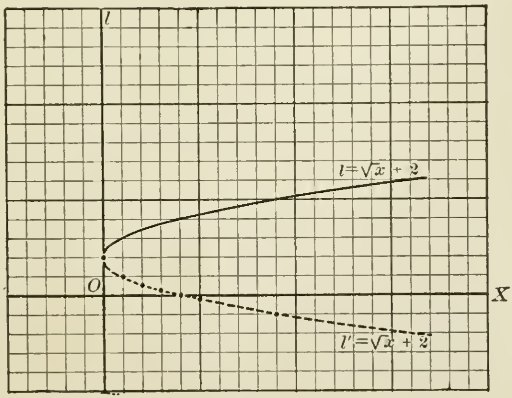


Fig. 67.

Now l' is actually zero when $x = 4$; it is then this second part of the figure that really corresponds to the solution given above.

In fact, the operation of *squaring both sides*, which we performed, has just the effect of bringing in the dotted portion of the figure; for when we squared both sides we got

$$x = 4.$$

If we now *try to go back to the preceding equation*, we should take the square root of both sides; this gives

$$\sqrt{x} = \pm 2,$$

where the sign $+$ has equal right with the sign $-$. Thus, instead of the original equation

$$\sqrt{x} = -2,$$

we find also a new one, $\sqrt{x} = +2,$

or, $\sqrt{x} - 2 = 0,$

whose left side is precisely l' above.

Squaring both sides of an equation is a process that cannot always be reversed; hence, as above, the result of doing so may lead to an incorrect answer because in connection with the original equation a new one is unconsciously introduced.

A sample of reasoning that is not reversible is the following:

"This object is a horse,
Hence, this object is an animal."

An attempt to reverse this leads to the absurdity:

"This object is an animal,
Hence, this object is a horse."

Such irreversible reasoning is dangerous in solving equations in algebra. Care should be taken to avoid such a process or where it is unavoidable to check the solution with care. While we have not before used any irreversible process, we have guarded against any unconscious errors by always checking the results.

146. Methods. The work to be done in solving simple radical equations is illustrated below.

Ex. 1. $\sqrt{2x+1}+7=x.$

Transpose 7: $\sqrt{2x+1}=x-7.$

Square both sides: $2x+1=x^2-14x+49.$

Transpose as indicated: $x^2-16x+48=0.$

Complete the square: $x^2-16x+64=16.$

Take the square roots: $x-8=\pm 4;$

hence, the only possible solutions are $x=8\pm 4=+4$ or $+12.$

Check: $x=+12$ gives $\sqrt{2\cdot 12+1}+7=12$ (correct),

hence, $x=+12$ is a solution.

$x=+4$ gives $\sqrt{2\cdot 4+1}+7=4$ (incorrect),

hence, $x=+4$ is *not* a solution.

Conclusion: the only solution of the given equation is $x=+12.$

Notice that $x=+4$ is a solution of the equation

$$-\sqrt{2x+1}+7=x,$$

which is formed by taking the *negative* radical $-\sqrt{2x+1}$ in place of the positive radical given.

Ex. 2. $\sqrt{2x+5}-\sqrt{x-1}=2.$

Transpose $\sqrt{x-1}$ and square both sides:

$$2x+5=4+(x-1)+4\sqrt{x-1}.$$

Transpose all except $4\sqrt{x-1}$ to one side:

$$4\sqrt{x-1}=x+2.$$

Square both sides: $16(x-1)=x^2+4x+4.$

Transpose as shown: $x^2-12x=-20.$

Complete the square: $x^2-12x+36=16.$

Take the square roots: $x-6=\pm 4,$

or, $x=6\pm 4=+10$ or $+2.$

Check: $x=+10;\ \sqrt{2\cdot 10+5}-\sqrt{10-1}=2$ (correct).

$x=+2;\ \sqrt{2\cdot 2+5}-\sqrt{2-1}=2$ (correct).

Conclusion: both $x=6$ and $x=2$ are solutions.

Notice that the equation

$$\sqrt{2x+5}+\sqrt{x-1}=2,$$

found by changing the sign of *one* radical, has *no* solution whatever, though the work would lead to the same answers as above except for the *check*, which would show that neither answer was correct.

The general directions for solving radical equations are:

(1) Place the most complicated radical on one side by itself; then square both sides.*

(2) Simplify as far as possible, and then repeat step (1) as often as necessary to clear of radicals.

(3) Solve the resulting equation if possible, being careful to check every answer.

(4) Discard every apparent answer that does not satisfy the given equation.

EXERCISES VIII: CHAPTER XI

1. $\sqrt{2x+3}-3=0.$

2. $\sqrt{3x-8}+2=0.$

3. $\sqrt{x-2}=14-x.$

4. $\sqrt{x-2}=x-14.$

5. $x+\sqrt{3x-2}=4.$

6. $2y+\sqrt{y^2+7}=-2.$

7. $a+\sqrt{2a-1}=8.$

8. $3v-\sqrt{v^2+3v-1}=3.$

9. $2x-1+\sqrt{2x^2-7}=12.$

10. $\sqrt{3p+1}+1=p.$

11. $2\sqrt{2z-3}=z-4.$

12. $\sqrt{k^2-3k-6}+8=2k.$

13. $\sqrt{2x^2+5x+2}+2=2.$

14. $\sqrt{3y^2+y+2}=y+2.$

15. $\sqrt{x}+\sqrt{x-7}=7.$

16. $\sqrt{p}+\sqrt{2p-2}=7.$

---

* In most of the exercises below, the radicals that occur are of the *second* degree. In exercises in which radicals of higher degree occur, raise both sides to the corresponding power.

17. $\sqrt{p+3}+\sqrt{p-2}=5.$

18. $\sqrt{3s+1}-\sqrt{2s-1}=1.$

19. $\sqrt{x^2+x+6}=\sqrt{x^2-9}.$

20. $\sqrt{x+1}+\sqrt{x-4}=\sqrt{x+17}.$

21. $\sqrt{3z-2}+\sqrt{z-1}-\sqrt{4z+1}=0.$

22. $\sqrt[3]{x^3+7}=x+1.$

23. $\sqrt[3]{x^2+x}-\sqrt[3]{x}=0.$

24. $\sqrt{x-3}+\sqrt{x+3}=\sqrt{2x}.$

25. $\dfrac{\sqrt{2x+5}+\sqrt{x-1}}{\sqrt{2x+5}-\sqrt{x-1}}=4.$

147. Square Root of Surds. The preceding exercises were especially selected so that the answers should not be *surd.* Otherwise we should have to find the *square root of a surd* in checking the problem.

Ex. 1. (1) $\sqrt{2x+4}-\sqrt{x-1}=2.$ (Compare Ex. 2, p. 303.)

Solving as above, the student will find

$$x=7\pm\sqrt{32}=7+4\sqrt{2} \text{ or } 7-4\sqrt{2}.$$

Check: $\sqrt{18+8\sqrt{2}}-\sqrt{6+4\sqrt{2}}=2$ (for $x=7+4\sqrt{2}$),

$\sqrt{18-8\sqrt{2}}-\sqrt{6-4\sqrt{2}}=2$ (for $x=7-4\sqrt{2}$).

We must then find, for example, $\sqrt{6+4\sqrt{2}}$.

The answer is $2+\sqrt{2}$,

for $$(2+\sqrt{2})^2=2^2+4\sqrt{2}+(\sqrt{2})^2=6+4\sqrt{2}.$$

But such an answer cannot be found by inspection. To find it directly suppose

(2) $$\sqrt{6+4\sqrt{2}}=\sqrt{r}+\sqrt{s},$$

where r and s are two unknown integers or fractions (*i.e.* rational numbers, hence not themselves surd, p. 284).

Squaring both sides, we get

(3) $$6+4\sqrt{2}=r+2\sqrt{rs}+s.$$

If we can find values of r and s so that

(4) $$r + s = 6,$$

and

(5) $$2\sqrt{rs} = 4\sqrt{2},$$

these values will satisfy (3), since (3) holds if (4) and (5) both hold. Square both sides of (4) and both sides of (5); then subtract; we find

(6) $$r^2 - 2rs + s^2 = 4,$$

or,

(7) $$r - s = \pm 2.$$

Solving (4) and (7) as linear simultaneous equations, we find $(r = 4, s = 2)$ or $(r = 2, s = 4)$; these satisfy (4) and (5), hence they satisfy (2), as will be found on trial. Substituting in (2), we have $\sqrt{6 + 4\sqrt{2}} = \sqrt{4} + \sqrt{2} = 2 + \sqrt{2}$, which is the value used above.

148. Equality of Surd Expressions. In § 147 we replaced equation (3) by equations (4) and (5). Let us show that this is surely correct.

Transpose $(r + s)$ in equation (3):

$$[6 - (r + s)] + 4\sqrt{2} = 2\sqrt{rs}.$$

Square both sides:

$$[6 - (r + s)]^2 + 2[6 - (r + s)]\cdot 4\sqrt{2} + 32 = 4rs.$$

We can now solve this equation for $\sqrt{2}$ *unless its coefficient* $[6 - (r + s)]$ is zero; if we could solve we should get

$$\sqrt{2} = \frac{4rs - [6 - (r + s)]^2 - 32}{8[6 - (r + s)]},$$

where the right side contains only rational expressions. *This is absurd*, for $\sqrt{2}$ is *not* rational. (See p. 184.) The only escape from this absurdity is that the coefficient $[6 - (r + s)]$ mentioned above should be zero:

$$6 - (r + s) = 0, \text{ or } r + s = 6,$$

which is (4) of § 147.

Subtracting this new equation from (3), we get

$$2\sqrt{rs} = 4\sqrt{2},$$

which is (4) of § 147.

This argument is another instance of *reduction to an absurdity* (reductio ad absurdum). See § 88, p. 166.

This justifies completely the work of § 147.

In general, by a similar argument, we may say that if

$$a + b\sqrt{c} = u + v\sqrt{w}$$

where a, b, c, u, v, w are rational (*i.e.* rational fractions or integers) and $\sqrt{c}$ is irrational, then $u = a$ and $v\sqrt{w} = b\sqrt{c}$, *i.e.* the rational terms are equal, and the irrational terms are equal. It will be found that some surds cannot have a square root of the form $\sqrt{r} + \sqrt{s}$ where r and s are rational. For example, the surd expression $\sqrt{9 + 2\sqrt{2}}$ will give no rational values of r and s.

EXERCISES IX: CHAPTER XI

Extract the following square roots; check by squaring the results:

1. $\sqrt{3 + 2\sqrt{2}}$.

2. $\sqrt{7 - 4\sqrt{3}}$.

3. $\sqrt{12 + 4\sqrt{8}}$.

4. $\sqrt{13 - 2\sqrt{42}}$.

5. $\sqrt{19 + 6\sqrt{10}}$.

6. $\sqrt{27 - 6\sqrt{20}}$.

7. $\sqrt{16 + 6\sqrt{7}}$.

8. $\sqrt{14 + 2\sqrt{13}}$.

9. $\sqrt{40 + 10\sqrt{15}}$.

10. $\sqrt{18 - 2\sqrt{72}}$.

11. $\sqrt{7 + 2\sqrt{10}}$.

12. $\sqrt{18 - 4\sqrt{20}}$.

13. $\sqrt{1 + 2\sqrt{a - a^2}}$.

14. $\sqrt{2p - 2\sqrt{p^2 - a^2}}$.

15. $\sqrt{2\left(m^2 + n^2 + \sqrt{m^4 + m^2n^2 + n^4}\right)}$.

Solve the following equations, and check each result:

16. $\sqrt{2x - 6} + \sqrt{27 - 4x} = 3$.

17. $\sqrt{6x + 2} + \sqrt{17 - 4x} = 5$.

18. $\sqrt{4x - 6} + \sqrt{2x - 3} = 1$.

19. $\sqrt{3x-10}-\sqrt{x-4}=2.$

20. $\sqrt{3x-10}+\sqrt{x-4}=2.$

21. $\sqrt{2x^2+2x+20}-\sqrt{2x^2+13}=1.$

22. $\sqrt{4x^2+2}+\sqrt{12-x^2}=5.$

23. $\sqrt{14+4x-x^2}+\sqrt{14x-x^2-21}=5.$

24. $\sqrt{x^2-2x+28}-\sqrt{x^2-6x+39}=1.$

25. $\sqrt{90-18x-5x^2}-\sqrt{34-2x-5x^2}=4.$

149. Radical Coefficients. Equations may involve radicals in their coefficients only; in that case we may solve directly.

Ex. 1. $(2+\sqrt{3})x-4=\sqrt{3}.$

Transpose 4: $(2+\sqrt{3})\quad x=4+\sqrt{3}.$

Divide by $2+\sqrt{3}$:
$$x=\frac{4+\sqrt{3}}{2+\sqrt{3}}$$
$$=\frac{(4+\sqrt{3})\times(2-\sqrt{3})}{(2+\sqrt{3})\times(2-\sqrt{3})}=\frac{8-2\sqrt{3}-3}{4-3}$$
$$=5-2\sqrt{3}.$$

Check:
$$(2+\sqrt{3})(5-2\sqrt{3})-4=\sqrt{3},$$
or,
$$(10+\sqrt{3}-2\times 3)-4=\sqrt{3}\text{ (correct).}$$

Ex. 2. $x^2+4x-2\sqrt{3}=0.$

Transpose $2\sqrt{3}$: $\quad x^2+4x=2\sqrt{3}.$

Complete the square: $\quad x^2+4x+4=4+2\sqrt{3}.$

Take square roots: $\quad x+2=\pm\sqrt{4+2\sqrt{3}}.$

or, $\quad x=-2\pm\sqrt{4+2\sqrt{3}}.$

$$\sqrt{4+2\sqrt{3}}=\sqrt{r}+\sqrt{s}.$$

Then,
$$r+s=4,$$
$$r-s=\pm 2.\quad \text{(See § 148.)}$$

Whence, $r = 3$ or 1 and $s = 1$ or 3.

$$\sqrt{4 + 2\sqrt{3}} = \sqrt{3} + \sqrt{1} = 1 + \sqrt{3};$$

hence, $x = -2 \pm (1 + \sqrt{3}),$

or, $x = -1 + \sqrt{3}$ or $-3 - \sqrt{3}$.

Check for $x = -1 + \sqrt{3}$: $(-1 + \sqrt{3})^2 + 4(-1 + \sqrt{3}) - 2\sqrt{3} = 0,$

or, $4 - 2\sqrt{3} - 4 + 4\sqrt{3} - 2\sqrt{3} = 0$ (correct).

Check for $x = -3 - \sqrt{3}$: $(-3 - \sqrt{3})^2 + 4(-3 - \sqrt{3}) - 2\sqrt{3} = 0,$

or, $12 + 6\sqrt{3} - 12 - 4\sqrt{3} - 2\sqrt{3} = 0$ (correct).

EXERCISES X: CHAPTER XI

Solve the following equations:

1. $(3 - \sqrt{2})x - 4 = \sqrt{2}.$

2. $(\sqrt{3} + \sqrt{2})x = \sqrt{6}.$

3. $(5 + \sqrt{3})x + 7 = 3\sqrt{3}.$

4. $4x + 1 = \sqrt{3}(x + 2).$

5. $x + 1 = \sqrt{2}(x - 3).$

6. $\dfrac{2 + x^2}{2 - x^2} = \sqrt{3}.$

7. $\dfrac{3 - x^2}{3 + x^2} = \dfrac{2}{\sqrt{7}}.$

8. $x^2 + (1 - \sqrt{3})x - 2(1 + \sqrt{3}) = 0.$

9. $x^2 - 5x + 2(\sqrt{2} - 1) = 0.$

10. $x^2 - 9x + (13 - 3\sqrt{5}) = 0.$

11. $x^2 - (5 - \sqrt{3})x - \sqrt{3} = 0.$

12. $x^2 - 3\sqrt{3} \cdot x + 4 + \sqrt{6} = 0.$

13. $x^2 - 5\sqrt{2} \cdot x + 5 - \sqrt{14} = 0.$

14. $x^2 - 3(\sqrt{5} - \sqrt{3})x + 16 - 5\sqrt{15} = 0.$

15. $(\sqrt{2} - 1)x^2 + (3\sqrt{2} - 6)x + (2\sqrt{2} - 1) = 0.$

16. $\begin{cases} \sqrt{3} \cdot x + \sqrt{2} \cdot y = 1, \\ \sqrt{2} \cdot x - \sqrt{3} \cdot y = 1. \end{cases}$

17. $\begin{cases} x+(3-\sqrt{5})y=4-\sqrt{5}, \\ 2x+(1+2\sqrt{5})y=2+3\sqrt{5}. \end{cases}$

18. $\begin{cases} x^2+y^2=20, \\ xy+x-y=-3. \end{cases}$

(SUGGESTION. Subtract twice the second equation from the first.)

19. $\begin{cases} x+y+xy=5, \\ x^2-y^2+8y=16. \end{cases}$

(SUGGESTION. Solve the second equation for x or y, and substitute in the first.)

20. $\begin{cases} xy+7x=46, \\ 7x^2+10xy=299. \end{cases}$

REVIEW EXERCISES XI: CHAPTER XI

Solve the equations:

1. $\sqrt{x-7}+\sqrt{2x-7}=8.$

2. $\sqrt{7x+2}-\sqrt{3x-2}=2.$

3. $\sqrt{x^2-3x-1}-\sqrt{x^2+3x+9}=-4.$

4. $x-\sqrt{4x^2-25x+6}=6.$

5. $\sqrt{x^2-2x-10}+\sqrt{x^2-6x-6}=6.$

6. $\sqrt{x^2-14x+55}-\sqrt{x^2+2x-17}=4.$

7. $\sqrt{2x^2-2x+11}-\sqrt{2x^2-4x+8}=1.$

8. $\sqrt{2x^2-4x-17}-\sqrt{2x^2-12x+15}=2.$

9. $\sqrt{4x+74-x^2}-\sqrt{24x-66-x^2}=2.$

10. $\sqrt{x^2+2}+\sqrt{4x^2+2}=3.$

11. $\sqrt{6x}-2\sqrt{8-2x}=\sqrt{48-10x}.$

12. $\sqrt{x+2}-\sqrt{2x-10}=\sqrt{3x-20}.$

13. $\sqrt{22-x^2}-\sqrt{2x^2+1}=6.$

14. $7x^2+2=\sqrt{3}(5x^2-6).$

15. $x^2-3x-(4+\sqrt{6})=0.$

16. $x^2-(4\sqrt{5}+6\sqrt{2})x+(1+4\sqrt{10})=0.$

17. $(2-\sqrt{3})x^2+(10-7\sqrt{3})x+(17-9\sqrt{3})=0.$

18. $\begin{cases}\sqrt{7}\cdot x-y=6,\\ 2x-3\sqrt{7}\cdot y=2.\end{cases}$

19. $\begin{cases}(\sqrt{2}+2\sqrt{3})x+(\sqrt{2}-\sqrt{3})y=4,\\ (2\sqrt{2}-6)x+(3-\sqrt{2})y=7.\end{cases}$

20. $\begin{cases}y-xy=1,\\ y^2-4xy=3.\end{cases}$

SUMMARY OF CHAPTER XI: RADICALS; FRACTIONAL AND NEGATIVE EXPONENTS; RADICAL EQUATIONS, pp. 284–311

PART I. OPERATIONS; FRACTIONAL AND NEGATIVE EXPONENTS. pp. 284–299.

Essential Rules: I. $(x^m)^n = x^{m \cdot n}$; II. $x^n \times x^m = x^{m+n}$; III. $x^n \times y^n = (x \cdot y)^n$; III *a.* $x^n \div y^n = \left(\frac{x}{y}\right)^n$; basis for extension; even roots of negative numbers — called imaginary — excluded. § 135, pp. 284–285.

Meaning of Fractional Exponents: extension under Rule I; $x^{\frac{p}{q}} = (\sqrt[q]{x})^p$; take positive answer if two exist. § 136, pp. 285–286.

Multiplication and Division, Radicals of Same Degree: equivalence of any root of a product to product of roots; reverses; division; removal of factors. Exercises I. § 137, pp. 286–289.

Addition, Similar Radicals: essentially, principle $a(b+c) = ab + ac$. reduction to similar radicals. Exercises II. § 138, pp. 289–290.

Reduction to Different Degree: essentially, fractional exponents obey laws of fractions. Exercises III. § 139, pp. 290–292.

Rationalization of Parts: monomial denominator — multipy by radical in denominator; quadratic binomial denominator — multipy by conjugate. Exercises IV and V. §§ 140–142, pp. 292–295.

Negative or Zero Exponent: $x^0 = 1$; $x^{-n} = \frac{1}{x^n}$; illustrative problems. Exercises VI. §§ 143–144, pp. 296–298.

Review Exercises, Part I, Chapter XI: Exercises VII. p. 299.

PART II. RADICAL EQUATIONS. pp. 300–311.

Generalities: possibility of elusive answers; graph; reversibility of steps necessary; check necessary. § 145, pp. 300–302.

Methods: clearing of radicals by throwing radicals to one side and raising to power. Exercises VIII. § 146, pp. 303–305.

Square Roots of Quadratic Surds: examples. § 147, pp. 305–306.

Equalities of Quadratic Surds: rational parts equal; surd parts equal; justification of § 147. Exercises IX. § 148, pp. 306–308.

Radical Coefficients: examples. Exercises X. § 149, pp. 308–310.

Review Exercises, Part II, Chapter XI: Exercises VII. pp. 310–311.

CHAPTER XII

EQUATIONS SOLVED BY SUBSTITUTION

150. Substitution. In the preceding Chapters we have solved a few examples (see pp. 178, 270) by substituting new letters in the place of inconveniently long expressions.

Thus, in squares of binomials we used

$$(x+y)^2 = x^2 + 2\,xy + y^2$$

as a standard (p. 93). Given a more complicated example, say $(4\,m^2n - 3\,nm)^2$, we *substituted* $4\,m^2n$ for x and $-3\,mn$ for y:

$$\begin{aligned}(4\,m^2n - 3\,mn)^2 &= (4\,m^2n)^2 + 2\,(4\,m^2n)(-3\,nm) + (-3\,nm)^2 \\ &= 16\,m^4n - 24\,m^3n^2 + 9\,n^2m^2\end{aligned}$$

by comparison with the formula above.

This process, which we have already used so often, is called **substitution.** Apparent answers are sometimes found that are not really roots (or solutions) of the given equations. The best possible safeguard is *to check the final answers* by direct substitution in the *given* equations.

151. Equations solved as Linear. Many equations may be solved by substitution more easily than by direct processes.

Ex. 1. $\dfrac{4}{x^2-2} + 3 = 5.$

Put $\quad s = \dfrac{1}{x^2-2}$; then $4\,s + 3 = 5$; whence, $s = \frac{1}{2}$,

or, $$\frac{1}{x^2-2} = \frac{1}{2}.$$

Clear of fractions: $\qquad x^2 - 2 = 2.$

Transpose: $\qquad x^2 = 4$, or, $x = \pm 2.$

Check: $x = \pm 2$ gives $\dfrac{4}{4-2} + 3 = 5$ (correct).

In this example the advantage of substituting a new letter is not very great; however, the principle illustrated by this simple example will be found useful below.

Ex. 2. $\begin{cases} \dfrac{2}{x-1} + \dfrac{10}{2y+3} = 3, \\ \dfrac{6}{x-1} + \dfrac{5}{2y+3} = 4. \end{cases}$

Put $\dfrac{1}{x-1} = u, \quad \dfrac{1}{2y+3} = v;$

then, $\begin{cases} 2u + 10v = 3, \\ 6u + 5v = 4, \end{cases}$

whence, $u = \frac{1}{2}, \quad v = \frac{1}{5}.$

Replacing the letters u and v by their meaning as above,

$$\frac{1}{x-1} = \frac{1}{2}, \quad \frac{1}{2y+3} = \frac{1}{5},$$

or, $x - 1 = 2, \quad 2y + 3 = 5;$

hence, $x = 3, \quad y = 1.$

Check: Setting $x = 3, \; y = 1$ in the given equation gives

$$\begin{cases} \frac{2}{2} + \frac{10}{5} = 3 \text{ (correct)}, \\ \frac{6}{2} + \frac{5}{5} = 4 \text{ (correct)}. \end{cases}$$

Ex. 3. $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = 0, \\ \dfrac{1}{x} - \dfrac{1}{y} = 0. \end{cases}$

Put $\dfrac{1}{x} = u, \quad \dfrac{1}{y} = v;$

then, $\begin{cases} u + v = 0, \\ u - v = 0, \end{cases}$

whence, $u = 0$ and $v = 0$ are the only possible solutions. But $u = 0$ gives $\dfrac{1}{x} = 0$, which gives no value for x. Likewise $v = 0$ gives no value for y. Hence,the given equations *have no solutions.*

EXERCISES I: CHAPTER XII

1. $\dfrac{8}{x^2-21}+5=7.$

2. $\dfrac{6}{5-2x^3}+3=5\left(4-\dfrac{9}{5-2x^3}\right).$

3. $\begin{cases} \dfrac{3}{x+y}+\dfrac{1}{x-y}=1, \\ \dfrac{10}{x+y}-\dfrac{3}{x-y}=-2. \end{cases}$

4. $\begin{cases} \dfrac{10}{r-s}+\dfrac{4}{2r-5s}=3, \\ \dfrac{5}{r-s}+\dfrac{8}{2r-5s}=3. \end{cases}$

5. $\begin{cases} \dfrac{1}{2p+3q}+\dfrac{4}{p-3q}=1, \\ \dfrac{7}{2p+3q}-\dfrac{2}{p-3q}=2. \end{cases}$

6. $\begin{cases} \dfrac{15}{r^2+s^2}-\dfrac{7}{r^2-s^2}=\dfrac{8}{5}, \\ \dfrac{50}{r^2+s^2}+\dfrac{21}{r^2-s^2}=-1. \end{cases}$

7. $\begin{cases} \dfrac{7}{2m^2-n^2}+\dfrac{1}{m^2-2n^2}=\dfrac{3}{2}, \\ \dfrac{14}{2m^2-n^2}+\dfrac{3}{m^2-2n^2}=4. \end{cases}$

8. $\begin{cases} x+y+\dfrac{1}{x-y}=10, \\ 3x+3y-\dfrac{10}{x-y}=17. \end{cases}$

9. $\begin{cases} 2u+3v-\dfrac{3}{5u-2v}=5, \\ 6u+9v+\dfrac{2}{5u-2v}=26. \end{cases}$

10. $\begin{cases} x^2+y^2+\dfrac{6}{x^2-y^2}=7, \\ 3x^2+3y^2-\dfrac{12}{x^2-y^2}=11. \end{cases}$

11. $\begin{cases} 2n^2-3t^2+\dfrac{3}{7t^2-3n^2}=9, \\ 6n^2-9t^2-\dfrac{13}{7t^2-3n^2}=5. \end{cases}$

12. $\begin{cases} \dfrac{1}{x+y}+x^2+y^2=62, \\ 2x^2+2y^2-\dfrac{50}{x+y}=72. \end{cases}$

13. $\begin{cases} \dfrac{15}{x^2+y^2-5}+\dfrac{10}{x+y-5}=11, \\ \dfrac{45}{x^2+y^2-5}-\dfrac{2}{x+y-5}=1. \end{cases}$

14. $\begin{cases} \dfrac{1}{y-4x}-x^2+y^2=25, \\ \dfrac{25}{y-4x}-2x^2+2y^2=73. \end{cases}$

15. $\begin{cases} 3x^2+3xy+3y^2-\dfrac{100}{x+y}=37, \\ x^2+xy+y^2+\dfrac{90}{x+y}=37. \end{cases}$

152. Equations solved as Quadratics. An equation may often be turned into a *quadratic* equation by substitution.

Ex. 1. $x^4 - 3x^2 + 2 = 0$.

Let $\qquad x^2 = s$; then $s^2 - 3s + 2 = 0$,

whence, $\qquad s = +1$ or $s = +2$

or, $\qquad x^2 = +1$ or $x^2 = +2$;

whence, $\qquad x = \pm 1$ or $x = \pm \sqrt{2}$.

Check: For $\qquad x = \pm 1$ gives $x^2 = 1$, $x^4 = 1$;

hence, $\qquad x^4 - 3x^2 + 2 = 1 - 3 + 2 = 0$ (correct).

For $\qquad x = \pm \sqrt{2}$ gives $x^2 = 2$, $x^4 = 4$;

hence, $\qquad x^4 - 3x^2 + 2 = 4 - 3 \cdot 2 + 2 = 0$ (correct).

This equation therefore has *four* roots: ± 1 and $\pm \sqrt{2}$. Draw the figure for $l = x^4 - 3x^2 + 2$ as on p. 204, note that the curve crosses the main horizontal line at points where

$$x = \pm 1 \text{ and } x = \pm \sqrt{2} = \pm 1.4 \text{ (about).}$$

Ex. 2. $x^4 - x^2 - 2 = 0$.

Let $\qquad x^2 = s$; then $s^2 - s - 2 = 0$,

whence, $\qquad s = 2$ or $s = -1$,

or, $\qquad x^2 = 2$ or $x^2 = -1$,

whence, $\qquad x = \pm \sqrt{2}$ or $x = \pm \sqrt{-1}$.

Since the result $\sqrt{-1}$ is meaningless (imaginary, see p. 211) for the present, we say that there are only two answers, $x = \pm \sqrt{2}$.

Check: $\qquad x = \pm \sqrt{2}$ gives $x^2 = 2$, $x^4 = 4$;

hence, $\qquad x^4 - x^2 - 2 = 4 - 2 - 2 = 0$ (correct).

The student may draw a figure as in example 1.

Ex. 3. $x^6 - 5x^3 + 4 = 0$.

Let $\qquad x^3 = s$: $s^2 - 5s + 4 = 0$,

whence, $\qquad s = 1$ or $s = 4$,

or, $\qquad x^3 = 1$ or $x^3 = 4$,

hence, $\qquad x = 1$ or $x = \sqrt[3]{4}$.

Check: if $x = 1$, $x^6 - 5x^3 + 4 = 1 - 5 \cdot 1 + 4 = 0$ (correct).

If $x^3 = 4$, $x^6 = 16$;

hence, $x = \sqrt[3]{4}$, $x^6 - 5x^3 + 4 = 16 - 5 \cdot 4 + 4 = 0$ (correct).

The roots 1 and $\sqrt[3]{4}$ found above are the only roots (except imaginary roots, which would be meaningless at present); this is clearly seen by drawing the figure, as in example 1.

EXERCISES II: CHAPTER XII

1. $x^4 - 7x^2 + 10 = 0.$

2. $t^4 - 3t^2 - 10 = 0.$

3. $p^4 - 5p^2 + 4 = 0.$

4. $4k^4 - 17k^2 + 4 = 0.$

5. $9x^4 - 10x^2 + 1 = 0.$

6. $3y^4 - 7y^2 + 2 = 0.$

7. $k^4 - 8k^2 + 4 = 0.$

8. $z^4 - 34z^2 + 1 = 0.$

9. $s^4 - 42s^2 + 9 = 0.$

10. $x^4 - 10x^2 + 1 = 0.$

11. $x^6 - 2x^3 + 1 = 0.$

12. $a^6 - 9a^3 + 8 = 0.$

13. $a^6 + 9a^3 + 8 = 0.$

14. $27c^6 + 28c^3 + 1 = 0.$

15. $8z^6 - 19z^3 - 27 = 0.$

16. $(x^2 + 2)^2 - 4(x^2 + 2) - 12 = 0.$

17. $\dfrac{6}{(x^2 - 3)^2} - \dfrac{7}{x^2 - 3} + 1 = 0.$

18. $(x^2 + 2x - 2)^2 - 3(x^2 + 2x - 2) - 18 = 0.$

19. $2x^4 - x^3 - 6x^2 - x + 2 = 0.$

Suggestion. Since $x = 0$ is not a solution, we may divide by x^2, with the result: $2x^2 - x - 6 - \frac{1}{x} + \frac{2}{x^2} = 0$. Now $x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2$. The substitution $x + \frac{1}{x} = u$ will be found useful in this case, and in all cases where the coefficients of terms equidistant from the two ends are the same.

20. $3x^4 + 4x^3 - 14x^2 + 4x + 3 = 0.$

153. Radical Equations. Equations containing radicals may often be solved easily by substitution.

Ex. 1. $x+3+\sqrt{x+3}=12.$

Let $\sqrt{x+3}=s$; then $s^2+s=12$,

whence, $s=+3$ or $s=-4$,

or, $\sqrt{x+3}=+3$ or $\sqrt{x+3}=-4$,*

whence, on squaring, $x+3=9$ or $x+3=16$,

or, $x=6$ or $x=13$.

Check: $x=6$ gives $6+3+\sqrt{6+3}=12$ (correct).

$x=13$ gives $13+3+\sqrt{13+3}=20$ (incorrect).

It follows that $x=6$ is a root, while $x=13$ is not a root. (See §§ 94 and 145, pp. 181 and 300.)

It is absolutely necessary to check each answer in such examples. (Compare p. 302.)

Ex. 2. $x^{\frac{2}{3}}-3x^{\frac{1}{3}}+2=0.$

Let $x^{\frac{1}{3}}=s$; then $s^2-3s+2=0$,

whence, $s=+1$ or $s=+2$,

or, $x^{\frac{1}{3}}=1$ or $x^{\frac{1}{3}}=2$;

hence, on cubing, $x=1^3=1$ or $x=2^3=8$.

Check: $x=1$ gives $1^{\frac{2}{3}}-3\cdot 1^{\frac{1}{3}}+2=1-3+2=0$ (correct).

$x=8$ gives $8^{\frac{2}{3}}-3\cdot 8^{\frac{1}{3}}+2=4-3\cdot 2+2=0$ (correct).

Hence, $x=1$ and $x=8$ are *both* roots of the given equation.

EXERCISES III: CHAPTER XII

1. $x+\sqrt{x}-6=0.$

2. $3t+\sqrt{t}-14=0.$

3. $z-2+7\sqrt{z-2}=30.$

4. $2k+7-20\sqrt{2k+7}+64=0.$

5. $10\sqrt{v+1}-\sqrt[4]{v+1}-24=0.$

6. $5\sqrt{n^2+3n}+n^2+3n=14.$

---

* We might reject this possibility at once, since $\sqrt{x+3}=-4$ contradicts our agreement that the sign $\sqrt{}$ denotes a *positive* answer. The possible solution $x=13$, which is derived below from this, is seen to be incorrect.

7. $8x^{\frac{2}{3}} + 19x^{\frac{1}{3}} - 27 = 0.$

8. $9u^{\frac{2}{5}} - 40u^{\frac{1}{5}} + 16 = 0.$

9. $9x^{\frac{4}{5}} - 22x^{\frac{2}{5}} + 8 = 0.$

10. $8(y^6 + 1) = 65y^3.$

11. $4x^{\frac{4}{5}} - 5x^{\frac{2}{5}} + 1 = 0.$

12. $r - 8\sqrt{2r + 7} + 11 = 0.$

[SUGGESTION. We may write $2r + 22 - 16\sqrt{2r + 7} = 0$, or $(2r + 7) - 16\sqrt{2r + 7} + 15 = 0$.]

13. $18m + 33 - 11\sqrt{3m + 5} = 0.$

14. $2r^3 + 1 - 11\sqrt{r^3 - 2} = 0.$

15. $x^2 + 7x + 1 - 4\sqrt{x^2 + 7x + 1} + 3 = 0.$

16. $2a^2 + 3a + 42 - 7\sqrt{4a^2 + 6a + 36} = 0.$

17. $p^2 + 3p - 16 + 2\sqrt{p^2 + 3p - 1} = 0.$

18. $9z^2 + 4z + 10 - 5\sqrt{9z^2 + 4z + 4} = 0.$

154. Simultaneous Equations. We can always solve simultaneous equations of which one is linear by the methods of §§ 86, 129. Simultaneous equations that may be solved as a pair of simultaneous linear equations have been mentioned in § 151. We note now that many forms may be reduced by substitution to one linear and one quadratic.

$$\text{Ex. 1.}\quad \begin{cases} \dfrac{1}{x^2} - \dfrac{1}{y^2} = 16, & (1) \\[2ex] \dfrac{1}{x} - \dfrac{1}{y} = 2. & (2) \end{cases}$$

Let $\frac{1}{x} = u$, $\frac{1}{y} = v$; the equations become

$$\begin{cases} u^2 - v^2 = 16, \\ u - v = 2, \end{cases}$$

which may be solved by the methods of § 129. We find $u = 5$, $v = 3$; hence, $x = \frac{1}{5}$, $y = \frac{1}{3}$. These answers are checked as follows:

Check:
$$\begin{cases} \dfrac{1}{(\frac{1}{5})^2} - \dfrac{1}{(\frac{1}{3})^2} = 25 - 9 = 16 \text{ (correct)}, \\ \dfrac{1}{\frac{1}{5}} - \dfrac{1}{\frac{1}{3}} = 5 - 3 = 2 \text{ (correct)}. \end{cases}$$

It is obvious, likewise, that examples may be given that can be written in the form of a pair of quadratic equations. These, however, lead to rather complicated work, unless very special examples are chosen.

Ex. 2. $\begin{cases} (x-2)^2 + (2y-3)^2 = 9, \\ (2y-3)^2 = 4(x-2) - 5. \end{cases}$

Write $x - 2 = u$, $2y - 3 = v$; then
$$\begin{cases} u^2 + v^2 = 16, \\ v^2 = 4u - 5. \end{cases}$$

Subtracting gives $\quad u^2 + 4u = 21,$

whence, $\quad u^2 + 4u + 4 = 25,$
$$u = -2 \pm 5 = -7 \text{ or } +3.$$

If $\quad u = -7,$
$$v^2 = 4u - 5 = -33,$$
and $\quad v = \sqrt{-33}$ (imaginary).

Hence, this leads to *no* real solution:

(*No solutions.*)

If $\quad u = +3,$
$$v^2 = 4u - 5 = 7,$$
$$v = \pm\sqrt{7}.$$

This gives two pairs of solutions:
$$\begin{cases} u = +3, \\ v = +\sqrt{7}. \end{cases} \quad \text{and} \quad \begin{cases} u = +3, \\ v = -\sqrt{7}. \end{cases}$$

Check: $\begin{cases} u = +3, \\ v = +\sqrt{7} \end{cases}$ give $\begin{cases} u^2 + v^2 (= 9 + 7) = 16, \\ v^2 (= 7) = 4u - 5 (= 12 - 5). \end{cases}$

And $\begin{cases} u = +3, \\ v = -\sqrt{7} \end{cases}$ give $\begin{cases} u^2 + v^2 (= 9 + 7) = 16, \\ v^2 (= 7) = 4u - 5 = (12 - 5). \end{cases}$

Corresponding to these, we find
$$x = u + 2 = 5,\ y = \frac{v+3}{2} = \frac{\pm\sqrt{7} + 3}{2},$$

which the student should check by substituting these values in the *given* equations.

EXERCISES IV: CHAPTER XII

1. $\begin{cases} \dfrac{1}{p^2} - \dfrac{1}{q^2} = 15, \\ \dfrac{1}{p} - \dfrac{1}{q} = 3. \end{cases}$

2. $\begin{cases} (x-5)^2 + (2y-3)^2 = 13, \\ (x-5) - (2y-3) = 1. \end{cases}$

3. $\begin{cases} y^6 + z^6 = 65, \\ y^3 + z^3 = 9. \end{cases}$

4. $\begin{cases} (r+s)(3r-2s) = 28, \\ (3r-2s)^2 - (r+s)^2 = 33. \end{cases}$

5. $\begin{cases} \dfrac{1}{uv} + \dfrac{1}{v} = 8, \\ \dfrac{1}{uv} + \dfrac{1}{v^2} = 10. \end{cases}$

6. $\begin{cases} \dfrac{x}{y} = 80, \\ x + \dfrac{1}{y} = 24. \end{cases}$

7. $\begin{cases} \dfrac{2}{(m-3)^2} = \dfrac{n+3}{m-3}, \\ \dfrac{1}{(m-3)^2} - 2(n+3)^2 = -28. \end{cases}$

8. $\begin{cases} \dfrac{1}{y} + \dfrac{3}{x} = 9, \\ \dfrac{1}{x^2} + \dfrac{1}{xy} = 10. \end{cases}$

9. $\begin{cases} u^2 + \dfrac{u}{v} = 21, \\ \dfrac{1}{v^2} + \dfrac{u}{v} = 28. \end{cases}$

10. $\begin{cases} \dfrac{p}{q} + \dfrac{q}{p^2} = \dfrac{17}{2}, \\ \dfrac{p^2}{q^2} + \dfrac{q}{p^2} = \dfrac{33}{4}. \end{cases}$

11. $\begin{cases} \dfrac{1}{x-y} + \dfrac{3}{x+y} = 10, \\ \dfrac{3}{(x-y)^2} - \dfrac{11}{(x+y)^2} = 4. \end{cases}$

12. $\begin{cases} \dfrac{x}{y-3} + \dfrac{y}{x-4} = 9, \\ \dfrac{x^2}{(y-3)^2} - \dfrac{y^2}{(x-4)^2} = 63. \end{cases}$

13. $\begin{cases} 2\,\dfrac{x^2+y^2}{xy} + x - y = 7, \\ 4\,\dfrac{x^2+y^2}{xy} + (x-y)^2 = 14. \end{cases}$

14. $\begin{cases} \dfrac{r^2-s^2}{4r-s} + \dfrac{s}{r-1} = 3, \\ \dfrac{(r^2-s^2)^2}{(4r-s)^2} + \dfrac{s^2}{(r-1)^2} = 5. \end{cases}$

SUMMARY OF CHAPTER XII: EQUATIONS SOLVED BY SUBSTITUTION, pp. 313–322

Substitution : new letters in place of longer combinations.
§ 150, p. 313.

Equations solved as Linear : illustrations. Exercises I.
§ 151, pp. 313–315.

Equations solved as Quadratic : illustrations. Exercises II.
§ 152, pp. 316–317.

Radical Equations : illustrations. Exercises III.
§ 153, pp. 318–319.

Simultaneous Equations : illustrations. Exercises IV.
§ 154, pp. 319–322.

CHAPTER XIII. PROGRESSIONS OR SEQUENCES

PART I. ARITHMETIC SEQUENCES

155. Sequences or Progressions. A set of numbers that follow one another in a definite order is called a **sequence** or a **progression**. Any one of the numbers is called a **term** or an **element** of the sequence.

In this Chapter only those sequences are treated in which there are a definite number of terms, so that the sequence finally stops. Such a sequence is called **finite**. Other sequences exist; for example, the integers 1, 2, 3, 4, etc., ⋯ form an **unending** or **infinite** sequence. (See Appendix, §§ 26–28.)

156. Arithmetic Progressions or Sequences. A sequence is called an **arithmetic progression** (or **arithmetic sequence**) if the difference between any term and the term which precedes it is always constant. This difference is called the **common difference**. We shall use the word *sequence* by preference from now on, instead of *progression*.

For example, the sequence of integers

$$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10$$

is an arithmetic sequence, for the difference between consecutive terms is one in all cases. Similarly, the sequences

(a)	$3,\quad 5,\quad 7,\quad 9,\quad 11,\quad 13$	(common difference 2)
(b)	$5,\quad 10,\quad 15,\quad 20$	(common difference 5)
(c)	$-4,\ -1,\ +2,\ +5,\ +8$	(common difference 3)
(d)	$7,\quad 3,\ -1,\ -5$	(common difference -4)
(e)	$-\frac{1}{2},\ +1,\ +\frac{5}{2},\ +4,\ +\frac{11}{2}$	(common difference $\frac{3}{2}$)

and, in general,

(*f*) $a,\ a + d,\ a + 2\,d$, etc., $\cdots,\ a + nd$

are all arithmetic sequences.

The first term will usually be denoted by a, the common difference by d, as in example (*f*).

If d is positive, the terms *increase;* in that case the sequence is called **increasing**. If d is *negative*, the terms *decrease;* the sequence is then called **decreasing**.

157. Any Term. Any term may be found by adding to the first term the common difference as many times as there are intervening steps; thus, the second term is $a + d$, the third is $a + 2\,d$, etc.; in general, the number of steps is one less than the number of the term: *hence the nth term is*

$$a + (n - 1)d,$$

or, $$t_n = a + (n - 1)d$$

where t_n means the nth term. If there are only n terms, the nth is the *last* one, and we write

(I) $$\boldsymbol{l = a + (n - 1)d,}$$

where l now denotes the last term.

Similarly, we may find the first term if we know the fourth (say) and the common difference: in this case, $a = t_4 - 3\,d$; or, in general:

$$a = t_n - (n - 1)d$$

where t_n means the nth term.

Ex. 1. If an arithmetic sequence starts with the number 4 and has a common difference 5, find the 10th term.

Here $a = 4,\ d = 5,\ n = 10$; hence, $t_{10} = 4 + 9 \cdot 5 = 49$.

Ex. 2. $a = -7\frac{1}{2},\ d = 2$, to find t_{12}.

Here $n = 12$; hence, $t_{12} = -7\frac{1}{2} + 12 \cdot 2 = 17\frac{1}{2}$.

Ex. 3. $a = +6,\ d = -4$, to find t_3.

$$t_3 = 6 + 3(-4) = -6.$$

158. The Sum. To find the *sum* of all the terms of an arithmetic sequence

$$s = a + (a + d) + (a + 2d) + \cdots + (l - d) + l,$$

where l denotes the *last term*, write this in the form

$$s = l + (l - d) + (l - 2d) + \cdots + (a + d) + a$$

and *add* these two equations; we get

$$2s = (a + l) + (a + l) + (a + l) + \cdots + (a + l) + (a + l),$$

the term $(a + l)$ occurring n times if there are n terms; hence,

$$2s = n(a + l),$$

or,

(II) $$s = \frac{n}{2}(a + l).$$

From I and II, if we know the values of any three of the quantities, a, n, d, l, s, the other two may be found. Thus, given a, s, d, to find n and l, we substitute the value of l from I into II:

$$s = \frac{n}{2}(a + a + (n - 1)d),$$

or,

(III) $$s = \frac{n}{2}(2a + (n - 1)d),$$

and so on.

When necessary, to avoid ambiguity, we denote the sum of n terms by s_n instead of s and write $s_n = \frac{n}{2}(a + l)$, etc.

Ex. 1. Given $a = 2$, $l = 14$, $n = 6$, to find s and d:

I becomes $14 = 2 + (6 - 1)d$, whence $d = \frac{12}{5}$.

II becomes $s = \frac{6}{2}(2 + 14)$, whence $s = 48$.

Check: The sequence is 2, $4\frac{2}{5}$, $6\frac{4}{5}$, $9\frac{1}{5}$, $11\frac{3}{5}$, 14; its sum is

$$2 + 4\tfrac{2}{5} + 6\tfrac{4}{5} + 9\tfrac{1}{5} + 11\tfrac{3}{5} + 14 = 48 \text{ (correct).}$$

Ex. 2. Given $d = 3,\ n = 4,\ s = -1.$

I becomes $l = a + (4 - 1)3$ or $l = a + 9.$

II becomes $-1 = \frac{4}{2}(a + l)$ or $2(a + l) = -1$;

whence, solving these simultaneous linear equations for a and l, we find

$$a = -\tfrac{19}{4},\ l = \tfrac{17}{4}.$$

Check: The sequence is

$$\frac{-19}{4},\ \frac{-7}{4},\ \frac{+5}{4},\ \frac{+17}{4};$$

then $s = \left\{\frac{-19}{4}\right\} + \left\{\frac{-7}{4}\right\} + \left\{\frac{5}{4}\right\} + \left\{\frac{17}{4}\right\} + \frac{-4}{4} = -1$, as given.

The general method is illustrated by the examples: first insert the given values in formulæ I and II; then solve the two resulting equations for the two unknown letters.

Since n must be an integer, only integral values of n are given in problems. If, in a problem in which n is to be found, the value is not an integer, the problem is impossible, *i.e.* no arithmetic sequence exists for that problem.

EXERCISES I: CHAPTER XIII

(The first examples should be checked as above by actually writing down the sequence.)

Find the quantities not given:

	a	d	n	l	s
1.	5	3	8		
2.	5	-3	8		
3.		$\frac{1}{2}$	17	10	
4.	16	-3		-2	
5.	-3			1	-9
6.	0	$\frac{1}{2}$	5		
7.		-1	7	-4	
8.	$\frac{2}{3}$	$\frac{1}{6}$		3	

	a	d	n	l	s
9.	15		19	69	
10.	7		5		65
11			5	1	-5
12.	-24		3	16	
13.	5			13	45
14.	6	-1		-10	
15.	-1		7		56
16.		$\frac{1}{10}$	16		60
17.	100			300	40,200
18.			6	1	-24
19.	-9		7	-3	
20.		-2	7		21

21. Show that $t_m - t_n = (m-n)d$.

22. Show that the sum of all the terms of the sequence from the mth to the nth inclusive is $(n-m+1)\dfrac{2a+(m+n-2)d}{2}$.

23. Given $a=-12$, $d=2$, $s=-40$, find n. Write out the sequence, and indicate why there are two values of n. Check the result of example 22, by finding the sum of all terms of this sequence from the sixth to the eighth inclusive.

24. Given $l=11$, $d=\frac{1}{2}$, $s=116$; find n and a.

25. Given $a=2$, $d=3$, $s=77$; find n and l. How many true solutions of the problem are there in this case?

PART II. GEOMETRIC SEQUENCES

159. Geometric Sequences. A sequence is a **geometric sequence** if the ratio of any term to the preceding one is always the same; this **common ratio** is an important number.

Thus, $1, 2, 4, 8, 16, 32, 64$

is a geometric sequence, the common ratio being 2. Also,

$3, 12, 48, 192$	(common ratio, 4),
$5, -10, +20, -40, +80$	(common ratio, -2),
$2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27},$	(common ratio, $\frac{1}{3}$),
and, in general, $a, ar, ar^2, ar^3, \cdots, ar^n$	(common ratio, r).

160. Formulæ. As in the case of arithmetic sequences, we may find two formulæ: one for the nth term, the other for the sum of n terms.

If the first term is called a and the common ratio r, the second term is ar, the third is ar^2, etc.; since we multiply by r at each step, it is clear that the nth term is

$$t_n = ar^{n-1},$$

where t_n means the nth term. If the last term is called l, we may write

(I) $$\boldsymbol{l = ar^{n-1}}$$

if there are just n terms.

Again, the *sum* is

(1) $$s = a + ar + ar^2 + ar^3 + \cdots + ar^{n-2} + ar^{n-1}.$$

Multiplying by $-r$, we find

(2) $$-rs = -ar - ar^2 - ar^3 - \cdots - ar^{n-2} - ar^{n-1} - ar^n,$$

whence, adding (1) and (2),

$$s - rs = a - ar^n = a(1 - r^n),$$

(II) $$s = a\left(\frac{1-r^n}{1-r}\right).$$

After this formula has been obtained, it is easy to see that it is correct, for, by actual division,

$$\left(\frac{1-r^n}{1-r}\right) = 1 + r + r^2 + \cdots + r^{n-1},$$

whence, on multiplying by a,

$$a\left(\frac{1-r^n}{1-r}\right) = a + ar + ar^2 + \cdots + ar^{n-1} = s.$$

These formulæ I and II may be used as were the formulæ for arithmetic sequences.

Ex. 1. Given $a = 3$, $r = 2$, $n = 7$, we have

$$l = 3 \cdot 2^6 = 192, \quad s = 3\left(\frac{1-2^7}{1-2}\right) = 3\left(\frac{-127}{-1}\right) = 3 \cdot 127 = 381.$$

Check: The sequence is 3, 6, 12, 24, 48, 96, 192; the last term is 192 (as found); the sum of the terms is

$$3 + 6 + 12 + \cdots + 192 = 381.$$

Ex. 2. Given $r = 3$, $n = 4$, $s = 120$.

II becomes $120 = a\left(\frac{1-3^4}{1-3}\right) = a\left(\frac{1-81}{-2}\right) = 40\,a$, whence, $a = \frac{120}{40} = 3$.

I becomes $$l = 3 \cdot 3^3 = 81.$$

Check: The sequence is 3, 9, 27, 81; it is easy to see that all the conditions of the problem, and the answers, hold good.

The same process is used in all cases: the known numbers are substituted for the corresponding letters in I and II; the resulting equations are then solved for the two unknown quantities.

Problems in which n is not given, but is to be found, are not given because the solution requires logarithms (see Chapter XIV); these problems are also not particularly valuable since n is by its nature *an integer;* finally, it is always easy to solve such a problem *by trial,* since only *integral* values of n need be tried.

EXERCISES II: CHAPTER XIII

Find the quantities not given:

	a	r	n	l	s
1.	2	3	5		
2.	16	$-\frac{1}{2}$	6		
3.		2	5		31
4.	4		3		19
5.		$\frac{1}{2}$	5	1	
6.	3	$\frac{1}{3}$	4		
7.	2		3	98	
8.	3		6	.00003	
9.		$\sqrt{3}$	5	1	
10.		$\frac{1}{10}$	5		111.11
11.		2	4	16	
12.		-2	4	16	
13.	16		5	81	
14.		-1	13		-1
15.	3	$\frac{2}{3}$	3		
16.		$-\frac{1}{2}$	5		$5\frac{1}{2}$
17.	1		4	27	
18.			3	80	105

SUGGESTION. Here $\frac{s}{l}=\frac{a(r^2+r+1)}{ar^2}=\frac{r^2+r+1}{r^2}$.

19. Show that $t_m-t_n=ar^{n-1}(r^{m-n}-1)$.

20. Show that the sum of all the terms of the sequence from the nth to the mth inclusive is

$$a\cdot\frac{r^m-r^{n-1}}{r-1}.$$

161. Theorems in Factoring. The formula II is also useful in factoring; dividing both sides of it by a:

$$(1) \qquad \frac{1-r^n}{1-r} = 1 + r + r^2 + \cdots + r^{n-1},$$

which may be used as a **type form** for factoring expressions of the form $1 - r^n$, as in Chapter IV, pp. 91–102.

Again, setting $r = \frac{y}{x}$ and clearing of fractions, we get

$$(2) \qquad \frac{x^n - y^n}{x - y} = x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + y^{n-1},$$

which is a **type form** for factoring expressions of the form $x^n - y^n$.

If in (1) we put $r = -s$, we get

$$(3) \quad \frac{1-s^n}{1+s} = 1 - s + s^2 - s^3 + \cdots - s^{n-1}, \textit{ if } n \textit{ is an even integer},$$

or,

$$(4) \quad \frac{1+s^n}{1+s} = 1 - s + s^2 - s^3 + \cdots + s^{n-1}, \textit{ if } n \textit{ is an odd integer}.$$

Setting $s = \frac{y}{x}$ in these, we find the following **type forms**:

$$(5) \quad \frac{x^n - y^n}{x+y} = x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \cdots - y^{n-1}, \text{ if } n \text{ is even},$$

or,

$$(6) \quad \frac{x^n + y^n}{x+y} = x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \cdots + y^{n-1}, \text{ if } n \text{ is odd}.$$

In English,

from (2) $x - y$ *is a factor of* $x^n - y^n$, *if* n *is any positive integer;*

from (5) $x + y$ *is a factor of* $x^n - y^n$, if n is even;

from (6) $x + y$ *is a factor of* $x^n + y^n$, if n is odd.

Comparing with the type forms of p. 101, we see that $x^2 - y^2$ falls under (2) and under (5); in fact $x^2 - y^2$ is divisible by $x - y$ and by $x + y$, as we already know. The

form $x^3 - y^3$ falls under (2) only; it is divisible by $x - y$, and the quotient is $x^2 + xy + y^2$.

Likewise,

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2) \text{ by (6) (see p. 101);}$$

$$x^4 - y^4 = (x - y)(x^3 + x^2y + xy^2 + y^3) \text{ by (2),}$$

or also, $\quad = (x + y)(x^3 - x^2y + xy^2 + y^3)$ by (5),

or also, $\quad = (x - y)(x + y)(x^2 + y^2)$ by (2) and (5),

$$x^5 - y^5 = (x - y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4);$$

$$x^5 + y^5 = (x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4),$$

and so on. Compare also Appendix, § 6.

Ex. 1. Factor $32\,a^5 - b^5$.

This compares to (2) if $x = 2\,a$, $y = b$, $n = 5$; hence,

$$32\,a^5 - b^5 = (2\,a - b)[(2\,a)^4 + (2\,a)^3b + (2\,a)^2b^2 + (2\,a)b^3 + b^4]$$
$$= (2\,a - b)(16\,a^4 + 8\,a^3b + 4\,a^2b^2 + 2\,ab^3 + b^4).$$

Ex. 2. Factor $81\,t^4 - s^8$.

This corresponds to (2) if $x = 3\,t$, $y = s^2$, $n = 4$; hence, $x - y = 3\,t - s^2$ is a factor. By (5) $x + y = 3\,t + s^2$ is also a factor; hence, $(3\,t - s^2)(3\,t + s^2)$ is also a factor, *i.e.* $9\,t^2 - s^4$ is a factor. This results also by taking $x = 9\,t^2$, $y = s^4$, $n = 2$, so that $x^2 - y^2 = 81\,t^4 - s^8$. From either we find $81\,t^4 - s^8 = (9\,t^2 - s^4)(9\,t^2 + s^4) = (3\,t - s^2)(3\,t + s^2)(9\,t^2 + s^4)$.

EXERCISES III: CHAPTER XIII

Factor the following expressions:

1. $x^6 - y^6$.

2. $1 + y^7$.

3. $x^7 - y^7$.

4. $1 - y^{12}$.

5. $x^{11} \pm y^{11}$.

6. $8\,m^3 + 27\,n^3$.

7. $81\,a^4 - 16\,b^4$.

8. $125\,a^6b^9 - 1$.

9. $1 + 1024\,x^5$.

10. $1 - 64\,x^6y^{12}$.

11. $32\,m^5 + 243\,n^{15}$.

12. $625\,x^4y^2 - r^6$.

SUMMARY OF CHAPTER XIII: SEQUENCES OR PROGRESSIONS, pp. 323–332

Sequence (or Progression): set of numbers in definite order.
§ 155, p. 323.

Arithmetic Sequence: common difference between terms.
§ 156, pp. 323–324.

Formulæ: $t_n = a + (n-1)d$: $s = \left(\frac{n}{2}\right)(a + l)$. Exercises I.
§ 157–158, pp. 324–327.

Geometric Sequence: constant common ratio. § 159, p. 328.

Formulæ: $t^n = ar^{n-1}$; $l = ar^{n-1}$; $s = \frac{a(1-r^n)}{(1-r)}$. Exercises II.
§ 160, pp. 328–331.

Factoring $x^m \pm y^n$: $x - y$ a factor of $x^n - y^n$; $x + y$ a factor of $x^n - y^n$ when n is even; of $x^n + y^n$ when n is odd. Exercises III.
§ 161, pp. 331–332.

CHAPTER XIV

LOGARITHMS

162. Introduction. We shall now study a method by which long numerical computations are greatly simplified. Before proceeding with this chapter, the student should review the entire subject of exponents. (See §§ 135–144.) The following examples may serve as a basis of this review.

EXERCISES I: CHAPTER XIV

1. Simplify the following as much as possible without performing any long numerical computations:

a. $x^3 \times x^2$. *b.* $127^5 \times 127^9$. *c.* $74^7 \times 74$. *d.* $3^{27} \times 2^{27}$; $7^8 \times 5^9$.

e. $3^{17} \times 3^{17}$. Can this be done in two ways?

f. $g^7 \div g^3$. *g.* $(a^3)^5$. *h.* $\sqrt[3]{a^{12}}$. *i.* $\sqrt[3]{a^3}$. *j.* $x^7 \div x^{12}$.

k. $24^9 \div 6^9$; $243^9 \div 81^8$. *l.* $127^5 \div 127^5$. Can this be done in two ways?

m. $\sqrt[3]{8^{17}}$. *n.* $\sqrt[3]{27^6}$. Can this be done in two ways?

2. Express without an exponent:

a. 8^{-2}. *b.* $8^{\frac{2}{3}}$. *c.* $8^{-\frac{2}{3}}$. *d.* 125^0. *e.* $(\frac{1}{12})^0$. *f.* $(\frac{1}{9})^{-\frac{2}{3}}$.

3. Sum up all the principles you have used in Exs. 1 and 2 by completing the following equations. Tell which of the above examples are special cases of each:

a. $x^{-m} =$ *c.* $x^0 =$ *e.* $x^m \div x^n =$ *g.* $\sqrt[n]{x^m} =$

b. $x^{\frac{m}{n}} =$ *d.* $x^m \cdot x^n =$ *f.* $(x^m)^n =$ *h.* $x^{-n} =$

4. State in words the laws expressed in Ex. 3. (See §§ 135, 136, 143.)

5. Complete the following table by giving all the powers of 16 at intervals of $\frac{1}{4}$ from -4 to $+4$.

$$
\begin{array}{ll}
16^0 = 1 & 16^0 = 1 \\
16^{\frac{1}{4}} = 2 & 16^{-\frac{1}{4}} = \frac{1}{2} \\
16^{\frac{2}{4}} = 4 & 16^{-\frac{1}{2}} = \frac{1}{4} \\
16^{\frac{3}{4}} = 8 & 16^{-\frac{3}{4}} = \frac{1}{8} \\
\cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \\
\cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \\
16^4 = 65536 & 16^{-4} = \frac{1}{65536}
\end{array}
$$

6. By means of the table in example 5 find the value, wherever it can be done without any long computation, of the following expressions:

a. $512 \times 128 =$ what?

Solution $512 \times 128 = 16^{\frac{9}{4}} \times 16^{\frac{7}{4}} = 16^{\frac{9}{4}+\frac{7}{4}} = 16^4 = 65536.$

b. $16384 \div 64.$

c. $\dfrac{512 \times 32768}{8192}.$

d. $\sqrt[7]{128 \times 16384^3}.$

e. $\sqrt[11]{(\frac{1}{4096})^{12} \times (8)^4}.$

f. $\left(\dfrac{16384}{2048} - \left[\dfrac{1}{263^3} \times 8192\right]\right).$

The preceding method alone is not completely successful in this case, because we cannot perform the subtraction by means of exponents.

g. Can the following operations be performed by means of your table? Why not? $\sqrt[7]{1024}$, $\sqrt[5]{572}$.

163. A Table of Exponents. Our table is not sufficiently complete for practical purposes, as was seen when we attempted to solve 6 *g*, above, by means of it. In order to make a table that will be of practical use in examples involving any number, it will be more convenient to use 10 as a basis instead of 16, as 10 is better suited to the

decimal system. Let the student complete the following table and make himself familiar with it:

$$10^0 = 1 \qquad\qquad 10^0 = 1$$
$$10^1 = 10 \qquad\qquad 10^{-1} = .1$$
$$10^2 = 100 \qquad\qquad 10^{-2} = .01$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \qquad\qquad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$10^6 = 1000000 \qquad\qquad 10^{-6} = .000001$$

We may use this table as we did the other in solving certain simple problems, but it has the same difficulty as the other, for such numbers as 273, 1772, etc., are not found in it; but it has an advantage over the other table in that it is easier to remember.

Let us now make a similar table containing many more numbers. To do this, let us make a graph of the table, in other words, a graph of the equation

$$n = 10^L.$$

The preceding table gives values of n and of L as follows:

Point in Fig. 68:						M	L	K	A	B	C				
L:			-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
n:			.0000001					.1	1	10	100				1000000

[Let the student complete the table.]

Since n can never become negative, we take the starting point near the left edge of the paper.

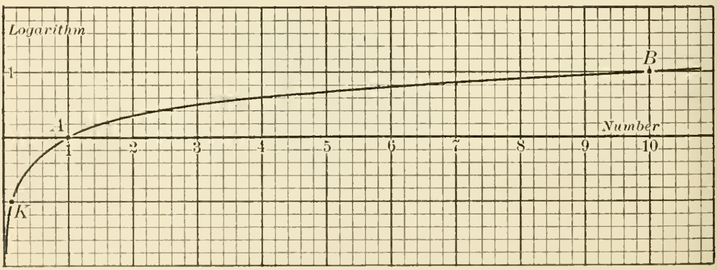


FIG. 68.

There is difficulty in making a graph of the whole table, owing to the large numbers that occur; moreover, we have not enough points to draw the curve very accurately; but the student will be able to answer the following questions:

How does n change as L becomes very large positively? How high does the curve rise? How does n change as L becomes very large negatively? Does it ever touch the vertical main line? How close does it get to it? How large a sheet of paper would be needed to plot all the points of the table? How finely would it need to be ruled?

Now let us draw the part of the curve from A to B carefully. Since the curve is so flat, we shall keep the same scale on the horizontal line, and *take the scale ten times as great on the vertical line;* and we shall take the starting point in the lower left-hand corner of our paper. We will try fractional values of L. Let $L = .5 = \frac{1}{2}$. Then $n = 10^{\frac{1}{2}} = \sqrt{10} = 3.16$. Let $L = .25 = \frac{1}{4}$. Then $n = 10^{\frac{1}{4}} = (10^{\frac{1}{2}})^{\frac{1}{2}} = (3.16)^{\frac{1}{2}} = \sqrt{3.16} = 1.78+$. Let $L = .75 = \frac{3}{4}$. Then $n = 10^{\frac{3}{4}} = 10^{\frac{1}{2}} \times 10^{\frac{1}{4}} = 3.16 \times 1.78 = 5.62$. Continuing this process, we find the values of n and L as given, in the following table, the computations being made in the order indicated by the letters of the alphabet.

Point in Fig. 69:	A	F	D	G	C	H	E	I	B
L (fractionally):	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{7}{8}$	1
L (decimally):	0.000	.125	.250	.375	.500	.625	.750	.875	1.000
n (to two places):	1.00	1.33	1.78	2.37	3.16	4.22	5.62	7.50	10.00

The points J, K, L, M, N, O, P, Q, in the figure correspond to values of L by sixteenths; the corresponding values of n may be computed by the student, or read off from the figure. The student should plot these values with great care, draw a smooth curve through them, and keep the figure for his own use in what follows. It should be like Fig. 69.

We can now determine by measurement from the figure the approximate value of the exponent L when the number n is given between 1 and 10; and of n when L is given between 0 and 1. The exponent L is often called the **logarithm** of n, and we write $L = \log n$. (See § 164.)

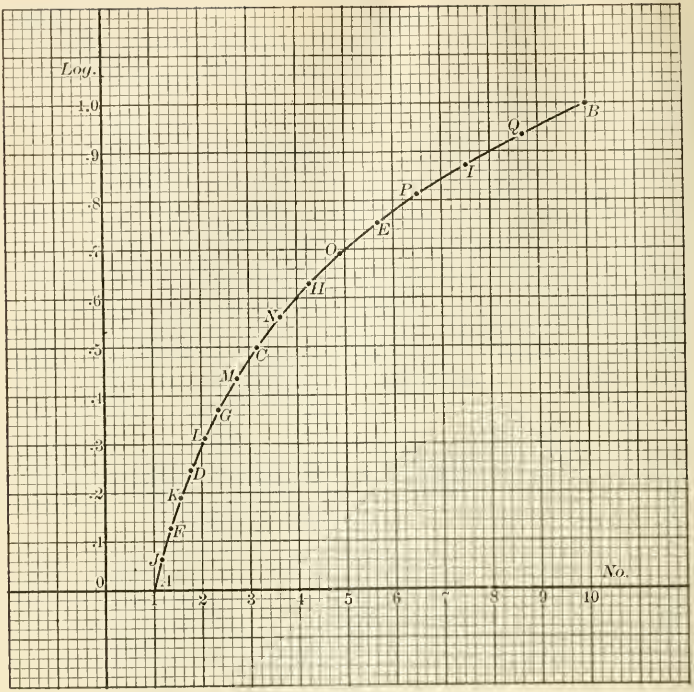


Fig. 69.

Ex. 1. Find L if $10^L = 4$.

We measure 4 to the right on the main horizontal line, and then the distance up to the curve is the required value, $L = .60$ (nearly). Hence, .60 is the logarithm of 4 (nearly), or $\log 4 = .60$ (nearly).

Ex. 2. Find $n = 10^{.53}$.

We measure .53 up on the main vertical line, and then the distance on a horizontal line to the curve is the required number, $n = 3.4$ (nearly). That is, $.53 = \log 3.4$ (nearly).

Continuing in a systematic manner, by actual measurement on the figure, as above, we find $\log 1 = 0$; $\log 1.1 = .04$; $\log 1.2 = .08$; $\log 1.3 = .11$.

These values can be conveniently arranged in tabular form, called a **table of exponents**, or a **table of logarithms**. In the table which follows, the values are correct to two places of decimals.

TWO PLACE TABLE

n	0	.1	.2	.3	.4	.5	.6	.7	.8	.9
1	0	.04	.08	.11	.15	.18	.20	.23	.26	.29
2	.30	.32	.34	.36	.38	.40		.43	.45	.46
3	.48	.49	.51	.52	.53		.56	.57	.58	.59
4	.60	.61	.62	.63	.64		.66	.67	.68	.69
5	.70	.71	.72		.73	.74	.75		.76	.77
6	.78		.79	.80		.81	.82		.83	.84
7		.85	.86	.86	.87		.88		.89	.90
8	.90	.91		.92		.93		.94		.95
9		.96		.97	.97	.98	.98		.99	
10	1									

[Let the student fill in the blank spaces from his figure. Notice that the value to be inserted must lie between those on either side of it. The values obtained by the student from his figure may not agree exactly with those given here, on account of the inaccuracy of drawing.]

To find from the table log 6.5 we look for the 6 in the column marked n; opposite 6 and in the column marked .5 we find .81, which is the logarithm of 6.5.

Values not actually in the table may also be found by a process of reasoning explained in the following examples:

Ex. 3. Find $n = 10^{2.53}$.

$n = 10^{2.53} = 10^2 \times 10^{.53} = 100 \times 3.39 = 339$, nearly; that is, $2.53 = \log 339$.

Ex. 4. Find L if $10^L = 7250$.

We find from the table that $7.25 = 10^{.86}$, nearly. But $1000 = 10^3$. Hence, $7250 = 1000 \times 7.25 = 10^3 \times 10^{.86} = 10^{3.86}$. Hence, $L = 3.86$. That is, $\log 7250 = 3.86$.

Ex. 5. Find log (.0725); that is, find L if $10^L = .0725$.

As above, $7.25 = 10^{.86}$, $.01 = 10^{-2}$. Hence, $.0725 = .01 \times 7.25 = 10^{-2} \times 10^{.86} = 10^{-2+.86}$. Hence, $\log .0725 = -2 + .86$. It is usually more convenient, when the integral part of the logarithm is negative, to leave the *decimal part always positive*, instead of adding; thus, we write $-2 + .86$ and *not* -1.14.

164. Simple Computation by Exponents. We can now perform any of the simple operations of multiplication, division, raising to powers, or extraction of roots upon any number whatever very easily by logarithms, though our results will not be precisely accurate.

Ex. 1. Multiply 2 by 3.

We can do this easily without logarithms: $2 \times 3 = 6$. Notice that $2 = 10^{.30}$, $3 = 10^{.48}$. Hence, $2 \times 3 = 10^{.30} \times 10^{.48} = 10^{.78} = 6$. Let the student make the measurements on his figure. The student's measurements may not agree precisely with these, owing to inaccuracy in the figure used.

Ex. 2. Multiply $3\frac{2}{3}$ by 7.3.

Let the student use his own figure. $3\frac{2}{3} = 10^{.56}$, $7.3 = 10^{.86}$. Hence, $3\frac{2}{3} \times 7.3 = 10^{1.42} = 10^1 \times 10^{.42} = 10 \times 2.6 = 26$. Multiplying by the ordinary method, we see that this is not precisely accurate, owing to the inaccuracy of our table, but this degree of accuracy would be sufficient in many practical problems.

Ex. 3. Divide 13 by 7.

$13 = 10 \times 1.3 = 10^1 \times 10^{.11} = 10^{1.11}$; $7 = 10^{.85}$. Hence, $13 \div 7 = 10^{1.11} \div 10^{.85} = 10^{.26} = 1.8$, nearly.

Ex. 4. Find $(4.3)^{17}$.

$4.3 = 10^{.63}$. Hence, $4.3^{17} = 10^{17 \times .63} = 10^{10.71} = 10^{10} \times 10^{.71} = 10000000000 \times 5.1 = 51000000000$. Of course, this is far from accu-

rate; we are sure only of the first one or two figures, but the student should notice the great saving in labor, and that frequently the accuracy here attained would be sufficient. With the more extended table given below, greater accuracy is obtainable.

Ex. 5. Extract the seventh root of 7825.

The work would be extremely long by methods previously used. We have here

$$7825 = 1000 \times 7.825 = 10^3 \times 10^{.89} = 10^{3.89}, \text{ nearly.}$$

Hence, $\sqrt[7]{7825} = \sqrt[7]{10^{3.89}} = 10^{.56} = 3.6$, nearly. In finding the logarithm of 7.825 by the table we took the nearest logarithm found in the table. A more accurate table will be found on p. 348.

EXERCISES II: CHAPTER XIV

Simplify the following by the aid of the preceding table; compare each result with the figure:

1. $4.3 \times 23.$ **4.** $\sqrt[13]{1730}.$ **7.** $45^2 + \sqrt[3]{(1.5)^2}.$

2. $230 \div 17.$ **5.** $2.3^7.$ **8.** $\left(\frac{240}{360}\right)^{\frac{7}{2}}.$

3. $2.7^{12}.$ **6.** $\sqrt[7]{2.3}.$ **9.** $.005^4 \times 5200^4.$

10. $\dfrac{\sqrt[3]{3.4} \times 4.1}{(.001)^5}.$ **11.** $\dfrac{(47)^{\frac{5}{2}}}{(35)^{\frac{2}{5}}}.$

165. Definitions and Principles. A fundamental number must always be chosen as the **base**; this is usually 10.

The **logarithm** *of a number to a given base is the exponent with which the base must be affected to produce the number.*

In other words $10^L = n$ and $\log n = L$ have precisely the same meaning. We have taken 10 as our base, as usual.

Henceforth, when the base is not specified, it will be understood that the base 10 is used. When the base 10

is used the logarithms are called **common logarithms** or **Briggs's logarithms.**

NOTE 1. The base 10 is most common, but others may be used. When necessary, we write $\log_{10} n$ to avoid ambiguity. In general, if the base is b, the logarithms are denoted by $\log_b n$. The base may be any number except 0 or 1.

NOTE 2. We have not explicitly defined the meaning of an irrational exponent, and the student should not be burdened at this stage with the idea. It is sufficient to call attention to the fact that the use of the *smooth curve* (§ 163) through certain points that can be located involves the essential idea of a rigorous treatment. (See Appendix, § 30.)

From the principles of exponents (§§ 104, 135, pp. 193, 285, and p. 334), it is evident that

I. *The logarithm of a product is equal to the sum of the logarithms of the factors*, for $10^m \times 10^n = 10^{m+n}$.

II. *The logarithm of a quotient is equal to the logarithm of the dividend less that of the divisor*, for $10^m \div 10^n = 10^{m-n}$.

III. *The logarithm of a power of a number is equal to the logarithm of the number multiplied by the exponent of the power*, for $(10^m)^n = 10^{mn}$. This principle includes the extraction of roots by using fractional exponents, for, by § 139,

$$(10^m)^{\frac{1}{n}} = 10^{m \times \frac{1}{n}} = 10^{\frac{m}{n}}.$$

These may be stated as follows:

I. $\text{Log}\,(m \times n) = \log m + \log n.$

II. $\text{Log}\,\dfrac{m}{n} = \log m - \log n.$

III. $\text{Log}\, n^K = K \log n$. ($K$ may be an integer or a fraction.)

The logarithm of 1 *to any base is zero.* Since $b^0 = 1$.

The logarithm of the base is 1. Since $b^1 = b$.

The common logarithm of a number between one and ten lies between zero and one. See the table or the figure.

166. Characteristic and Mantissa. Let us now study the curve $10^l = n$, or $\log n = l$ for a greater range of values. From the table already constructed (§ 163, p. 339) we find, for example, $10^{.53} = 3.4$. Hence,

$$
\begin{array}{ll}
10^{1.53} = 34. & 10^{-1+.53} = .34 \\
10^{2.53} = 340. & 10^{-2+.53} = .034 \\
10^{3.53} = 3400. & 10^{-3+.53} = .0034 \\
10^{4.53} = 34000. & 10^{-4+.53} = .00034 \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot
\end{array}
$$

The student should notice that *increasing the logarithm by* **1** *corresponds to multiplying the number by* **ten** (*i.e.* the decimal point is changed one place). Likewise, *decreasing* the logarithm by 1 corresponds to *dividing* the number by ten.

$$
\begin{array}{llll}
\text{If} & 1 < x < 10, & \log x = & 0 + \text{a positive fraction.} \\
\text{If} & 10 < x < 100, & \log x = & 1 + \text{a positive fraction.} \\
\text{If} & 100 < x < 1000, & \log x = & 2 + \text{a positive fraction.} \\
\cdot & \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot \\
\text{If} & .1 < x < 1, & \log x = & -1 + \text{a positive fraction.} \\
\text{If} & .01 < x < .1, & \log x = & -2 + \text{a positive fraction.} \\
\cdot & \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot
\end{array}
$$

The fractional part will *be the same* in all these cases *if the digits in the number, x, are the same and follow each other in the same order. The position of the decimal point* determines only *the integral part* of the logarithm.

The fractional part is called the **mantissa**, and is generally taken positive to avoid the introduction of a different decimal; it is determined by the digits in the number as given and does not depend on the position of the decimal point. A table is needed to find the value quickly.

The integral part is called the **characteristic**; it may be either positive or negative. This integral part can always

be found by inspection, as above; hence, a table of logarithms contains only the fractional parts of the logarithms.

Let us now change our scale, using $1 = 5$ small spaces on the vertical line and $1 = 1$ small space on the horizontal line. Plot a sufficient number of points by means of the table on p. 339 and the principle just given. Then on the same sheet plot the characteristic. This gives the stair step bounding the shaded region. The distance from the main horizontal line *up* or *down* to the stair step is the *characteristic*, the distance from the stair step *up* to the curve is the *mantissa*.

In making a table of logarithms it is customary to give only the mantissa, omitting the decimal point both in the number and in the logarithm, as mentioned above. The student is left to determine the characteristic. This is the same as giving the logarithms of numbers between one and ten.

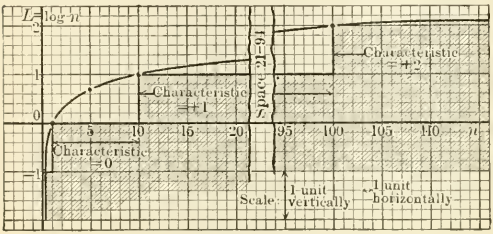


Fig. 70.

167. Four-place Table of Logarithms. The table on pp. 348–349 is constructed on the same plan as that on p. 339. The logarithms are given to four places of decimals and the corresponding numbers are given to three figures. If greater accuracy is required, tables can be bought to five, six, seven, or ten places. The logarithms as given in the following table cannot generally be exact, as all figures after the fourth place are rejected. If the fifth figure is 5 or more, the fourth figure is increased by one.

Thus, if the logarithm of a number is .437826 ···, it is given in the table as .4378 If the logarithm is .43455 ···, it is given as .4346

To find the Logarithm of a Given Number from the Table.

Ex. 1. Find the logarithm of 3.

In the columns marked N we look for 30 (the decimal point after the 3 is omitted). On a line with this in the column headed 0 we find 4771. Hence, $\log 3 = .4771$

Ex. 2. Find the logarithm of 4.6

In the column marked N we find 46. On a line with it in the column marked 0 we find 6628. Hence, $\log 4.6 = .6628$

Ex. 3. Find the logarithm of 3.76

In the column marked N we find 37. On a line with this in the column marked 6 we find 5752. Hence, $\log 3.76 = .5752$

Ex. 4. Find the logarithm of 3760.

$$\left.\begin{array}{ll}\text{By example 3,} & 10^{.5752} = 3.76.\\ \text{But,} & 10^{3} = 1000.\end{array}\right\} \quad \begin{array}{l}\text{Hence, } 10^{3.5752} = 3760\\ \text{or, } \log 3760 = 3.5752\end{array}$$

Ex. 5. Find the logarithm of .0376.

$$\left.\begin{array}{ll}\text{By example 3,} & 10^{.5752} = 3.76\\ \text{But,} & 10^{-2} = .01\end{array}\right\} \quad \begin{array}{l}\text{Hence, } 10^{-2+.5752} = .0376\\ \text{or, } \log .0376 = -2 + .5752\end{array}$$

This is sometimes written $\log .0376 = 8.5752 - 10$.

Ex. 6. Find log 3764.

From the table

$\log 3.76 = .5752$ (A)

$\log 3.77 = .5763$ (B)

We will plot these points on a very large scale, drawing only a small part of the curve.

A is the point where $n = 3.76$, $L = .5752$

B is the point where $n = 3.77$, $L = .5763$

E is the point where $n = 3.764$

We wish to find L.

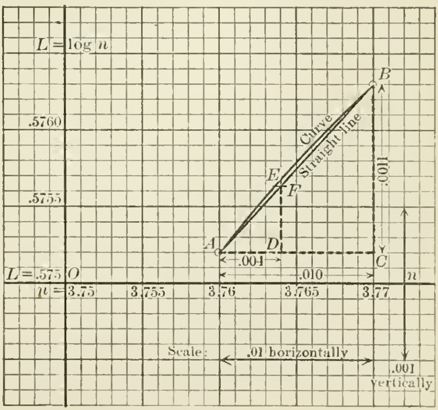


Fig. 71.

If we draw the straight line AB, cutting DE at F, then F will lie very near to E and it will be sufficiently accurate to find the height of F instead of E. D is just as high as A, *i.e.* .5752; $CB = .0011$; $DF = .4$ of $.0011 = .0004$ (dropping decimal places after the fourth).

Hence, the height of F is $.5752 + .0004 = .5756$

That is, $10^{.5756} = 3.764$

$10^3 = 1000.$

Hence, $10^{3.5756} = 3764.$

Hence, $\log 3764 = 3.5756$

This process of finding a result not given explicitly in the table is called **interpolation.**

Ex. 7. Find the logarithm of .02756

Consider 2.756

$$\begin{aligned} \log 2.75 &= .4393 \\ \log 2.76 &= \underline{.4409} \\ & .0016 \end{aligned}$$

$\log 2.756 = .4393 + (.6 \times .0016) = .4393 + .0010$ (nearly) $= .4403$

$10^{.4403} = 2.756$ Hence, $10^{-2+.4403} = .02756$

Hence, $\log .02756 = -2 + .4403 = 8.4403 - 10$. The student should draw a figure illustrating this work.

To find the Number corresponding to a given Logarithm.

When the decimal part of the logarithm can be found in the table, the corresponding number can be written down at once.

Ex. 1. Find n if $\log n = 4.9175$

Looking in the table, we find 9175 on a line with 82 (in column N) and in the column marked 7. Hence, $10^{.9175} = 8.27$ Hence, $10^{4.9175} = 82{,}700$. Hence, $\log 82{,}700 = 4.9175$ When the decimal part of the logarithm cannot be found in the table, we follow a process similar to that in examples 6 and 7 above.

Ex. 2. Find n if $\log n = 2.4574$

From the table we find

$$\begin{aligned} \log 2.87 &= .4579 & \log n &= 2.4574 \\ \log 2.86 &= .4564 & \log 286 &= 2.4564 \\ \text{Difference} &= .0015 & &= .0010 \end{aligned}$$

The difference between log 286 and log n = .0010 This is $\frac{10}{15}$ of the difference between log 286 and log 287. Hence, .4574 = log $(2.86 + \frac{10}{15}$ of .01). Hence, 2.4574 = log 286.7 The student should construct a figure like that above for this case.

Notice that the *differences* found above are given in the column marked D in the table. These differences are not always exact, since the real differences often vary in the same row; but they are always sufficiently accurate for use with this table.

EXERCISES III: CHAPTER XIV

1. Find the logarithm of:

(*a*) 7	(*c*) 600	(*e*) .032	(*g*) .0467	(*i*) 2.473
(*b*) 40	(*d*) 4.7	(*f*) 2.43	(*h*) 57,200	(*j*) .04257

2. Find the number whose logarithm is:

(*a*) .3010	(*c*) $\bar{2}.3802$	(*e*) 5.4533	(*g*) 2.6290	(*i*) 2.0027
(*b*) 3.4771	(*d*) $\bar{3}.3075$	(*f*) 1.5732	(*h*) $\bar{1}.4563$	(*j*) .0317

168. Computation by Logarithms.

Ex. 1. Simplify $\sqrt[3]{\frac{247 \times 3.428}{(16.3)^2}}$.

$$\sqrt[3]{\frac{247 \times 3.428}{(16.3)^2}} = \sqrt[3]{\frac{10^{2.3927} \times 10^{.5350}}{(10^{1.2122})^2}} = \sqrt[3]{\frac{10^{2.9277}}{10^{2.4244}}}$$

$$= \sqrt[3]{10^{.5033}} = 10^{.1677} = 1.471$$

This work may be tabulated as follows:

$$\log \sqrt[3]{\frac{247 \times 3.428}{(16.3)^2}} = \frac{1}{3}[\log 247 + \log 3.428 - 2 \log 16.3].$$

	log 247 = 2.3927	
	log 3.428 = .5350	
	log numerator =	2.9277
	log 16.3 = 1.2122	
	log $(16.3)^2$ =	2.4244
	log quotient =	.5033
Divide by 3:		
	log answer =	.1678
hence,	answer = 1.471	

N	0	1	2	3	4	5	6	7	8	9	D
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	42
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	38
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	35
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	32
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	30
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	28
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	26
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	25
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	24
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	22
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	21
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	20
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	19
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	18
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	18
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	17
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	16
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	16
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	15
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	15
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	14
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	14
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	13
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	13
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	13
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	12
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	12
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	12
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	11
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	11
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	11
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	10
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	10
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	10
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	10
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	10
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	9
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	9
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	9
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	9
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	8
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	8
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	8

N	0	1	2	3	4	5	6	7	8	9	D
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	8
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	8
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	8
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	7
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	7
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	7
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	7
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	7
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	7
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	7
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	6
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	6
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	6
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	6
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	6
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	6
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	6
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	6
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	5
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	5
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	5
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	5
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	5
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	5
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	5
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	5
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	5
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	5
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	4

Ex. 2. Simplify $\dfrac{\sqrt[15]{43+5\sqrt[3]{278}}}{\sqrt[5]{-17}}$.

The operations of addition and subtraction cannot be performed by logarithms. We may find $5\sqrt[3]{278}$ by logarithms, add this to 43, and then perform the remaining operations by logarithms. We handle the problem as if all the numbers were positive and then put the proper sign before the answer, which in this case is negative.

Ex. 3. Find $\sqrt[3]{.0247}$

$$\begin{aligned}\text{Log } \sqrt[3]{.0247} &= \tfrac{1}{3}\log .0247 = \tfrac{1}{3}(-2+.3927)\\ &= (-.6667+.1309) = -.5358\end{aligned}$$

We cannot find this in the table, since it is negative. Hence, we write in another form:

$$\begin{aligned}\log \sqrt[3]{.0247} &= \tfrac{1}{3}(-2+.3927) = \tfrac{1}{3}(8.3927-10)\\ &= (2.7976-3.3333) = 2.4643-3 = \bar{1}.4643\end{aligned}$$

We may do this in another and simpler way. By subtracting 1 from the -2, and adding 1 to the mantissa, we obtain $\frac{1}{3}(-3+1.3927) = \bar{1}+.4642$ Hence,

$$\log \sqrt[3]{.0247} = \bar{1}.4642, \text{ whence } \sqrt[3]{.0247} = .2912$$

Ex. 4. Solve $3^x = 17$.

We might put $L = 3^x$ and try various values of x and plot a curve.

x	1	2	3	$\frac{5}{2}$	$\frac{11}{4}$	$\frac{21}{8}$	$\frac{41}{16}$
L	3	9	27	15.59	20.51		

Thus, we would find an approximate value without logarithms, but the computation is very long. It can be done much more simply by logarithms.

$$3^x = 17$$

$$\log 3^x = \log 17$$

$$x \log 3 = \log 17, \text{ by III, § 165}$$

$$x = \frac{\log 17}{\log 3} = \frac{1.2304}{.4771} = 2.579$$

EXERCISES IV: CHAPTER XIV

Compute by the use of logarithms:

1. 43×75.

2. 437×9.63.

3. $439 \times .0372$,

4. $243.7 \times .179.2$.

5. $.8752 \times .01529$.

6. 1.002×3.075.

7. $20 \div 685$.

8. $.257 \div 1.73$.

9. $.0024 \div 1.034$.

10. $10.07 \div 4.617$.

11. $25{,}680 \div 152{,}980$.

12. $3 \div .002463$.

13. $47 \times 6.3 \times 250$.

14. $246 \times .0072 \times 102$.

15. $76 \times 50.04 \times .06004$.

16. $1205 \times 6\frac{2}{3} \times 54\frac{3}{7}$.

17. $24\frac{1}{2} \times 17\frac{3}{5} \times 246\frac{1}{3}$.

18. $(1.47)^3$.

19. $(3.057)^7$.

20. $(99.43)^5$.

21. $(1.01)^{25}$.

22. $(1.04)^{1800}$.

23. $\sqrt{64.39}$.

24. $\sqrt{.005726}$.

25. $\sqrt[3]{125.3}$.

26. $\sqrt[4]{1.072}$.

27. $\sqrt[27]{200}$.

28. $\sqrt[5]{.005}$.

29. $\sqrt[3]{\frac{3}{5}}$.

30. $\sqrt[7]{\frac{1}{8}}$.

31. $\dfrac{483 \times .035 - 2.461}{5 \times 4.6 \times 3547}$.

32. $\dfrac{642 \times 3729 \times .0007}{1705 \times 2.004 \times .4}$.

33. $\dfrac{546 \times .0001 \times .4040}{2.02 \times .003 \times .2623}$.

34. $\sqrt{\dfrac{46 \times 3.5}{94 \times 102}}$.

35 $\sqrt[2]{\dfrac{1247 \times 2391}{14.62 \times 24}}$.

36. $\sqrt[3]{\dfrac{5 \times 24 \times 3.1}{2 \times 18 \times 640}}$.

37. $\sqrt[4]{\frac{2}{41} \times \frac{3}{19} \times 240}$.

38. $\sqrt[5]{19.4 \times \dfrac{24}{976} \times \dfrac{1.72}{3.95}}$.

39. $\sqrt[16]{\dfrac{947 \times .3246}{89.32 \times 34.95}}$.

40. $\dfrac{5020\sqrt{.00437}}{4\sqrt[3]{97.3}}$.

41. $\dfrac{16\sqrt[5]{17 \times 43}}{247\sqrt[4]{5462}}$.

42. $\dfrac{(14)^{\frac{2}{3}} \times (129)^{\frac{4}{5}}}{(3.49)^{\frac{1}{3}}}$.

43. $\dfrac{(4.007)^{\frac{3}{4}} \times (.003^{\frac{2}{5}})}{\sqrt[7]{4263}}$.

44. $(\frac{4}{5})^3 \sqrt[16]{\frac{243}{17}}$.

45. $\sqrt[7]{.38\sqrt[2]{95}}$.

46. $\frac{3}{745}\sqrt[17]{\frac{20}{380}\sqrt[3]{475}}$.

47. $\sqrt[3]{16 - \sqrt[4]{92}}$.

48. $\dfrac{\sqrt{16 \times \sqrt{92}}}{\sqrt[3]{16 - \sqrt{92}}}$.

49. $\sqrt[6]{\dfrac{247 \times \sqrt{44}}{392 - \sqrt{44}}}$.

50. $\dfrac{-31 + \sqrt{(31)^2 - 4.3117}}{2.31}$.

51. $\dfrac{52 - \sqrt{52^2 - 4.6.19}}{2.6}$.

Solve examples 52 to 59 for x.

52. $3^x = 12$.

53. $9^x = 7$.

54. $17^{\frac{x}{2}} = \frac{2}{3}$.

55. $14^{3x} = 27$.

56. $16^{2x-3} = 7$.

57. $2^{x^2} = 47$.

58. $247 = \dfrac{5.4^x - 5}{6 - 1}$.

59. $256 = 4.7^{x-1}$.

60. Find the compound amount of \$ 100 for 15 years at 4 %.

61. Find the compound amount of \$ 1 for 100 years at 6 %.

62. What amount put at compound interest at 4 % at a child's birth will amount to \$ 1000 when he is 21 years of age?

63. At what rate at compound interest will \$ 1 amount to \$ 10 in 40 years?

64. In how many years will \$ 1 amount to \$ 2.00 at 4 % compound interest?

SUMMARY OF CHAPTER XIV: LOGARITHMS, pp. 334–352

Review of Exponents: restatements. Exercises I. § 162, pp. 334–335.

Temporary Table: logarithms as exponents of a chosen base; figure for $n = 10^L$; short table from figure and computation. — base 10. § 163, pp. 335–340.

Simple Computation by Exponents: illustrations of use of exponent laws. Exercises II. § 164, pp. 340–341.

Formal Definitions: base, — the number whose exponents are used; logarithm, — the exponents that produce given numbers; equivalence of $n = 10^L$ and $\log n = L$.

Formal Rules: logarithm of product, — sum of logarithms; logarithm of quotient, — difference of logarithms; logarithm of power, — power times logarithm. § 165, pp. 341–342.

Characteristic: integral part; increase of 1 for every digit place; judgment of characteristic without table.

Mantissa: fractional part; independent of decimal point; tables necessary. § 166, pp. 343–344.

Four Place Table: logarithm of given number; number for given logarithm; interpolation. Exercises III. § 167, 344–347.

Computations: illustrative examples; tables given (pp. 348–349). Exercises IV. § 168, pp. 347–352.

APPENDIX

NOTE I. DETACHED COEFFICIENTS

1. Detached Coefficients. We may considerably shorten the labor of many operations in *polynomials* by writing only the coefficients, taken in their natural order after the polynomials are arranged in ascending or descending powers of some one letter.

2. Multiplication.

Thus, to multiply $3x^3 - 7x^2 + 5x - 6$ by $2x^2 - 4x + 3$, we merely write

$$\begin{array}{rrrrrr}
3 & -7 & +5 & -6 & & \\
2 & -4 & +3 & & & \\
\hline
6 & -14 & +10 & -12 & & \\
 & -12 & +28 & -20 & +24 & \\
 & & 9 & -21 & +15 & -18 \\
\hline
6 & -26 & +47 & -53 & +39 & -18
\end{array}$$

and write the product

$$6x^5 - 26x^4 + 47x^3 - 53x^2 + 39x - 18,$$

the *power* of x which belongs in each term being clearly indicated by the position of the term.

3. Division.

Likewise, if we are to divide

$$6x^5 - 26x^4 + 47x^3 - 53x^2 + 39x - 18 \text{ by } 2x^2 - 4x + 3,$$

we write only the coefficients, as follows:

$$
\begin{array}{llll}
\textit{Dividend:} & 6 - 26 + 47 - 53 + 39 - 18 & \underline{|2 - 4 + 3} & \textit{Divisor} \\
& \underline{6 - 12 + \ \ 9} & |\underline{3 - 7 + 5 - 6} & \textit{Quotient} \\
& \quad - 14 + 38 - 53 & & \\
& \quad \underline{- 14 + 28 - 21} & & \\
& \qquad\qquad 10 - 32 + 39 & & \\
& \qquad\qquad \underline{10 - 20 + 15} & & \\
& \qquad\qquad\quad - 12 + 24 - 18 & & \\
& \qquad\qquad\quad \underline{- 12 + 24 - 18} & & \\
& \qquad\qquad\qquad\qquad\quad 0 & \textit{Remainder} &
\end{array}
$$

Whence, the quotient is $3x^3 - 7x^2 + 5x - 6$. Compare with the multiplication performed above.

The student must be extremely careful **not to omit** terms *even when the coefficient is zero*, for the position of the term indicates the degree.

Thus, $x^3 + 2x - 5$ should be written $1 + 0 + 2 - 5$, not $1 + 2 - 5$, for $1 + 2 - 5$ would represent $x^2 + 2x - 5$.

4. Division by $x - a$. Division by a simple binomial of the form $(x - a)$ is especially easy.

Let us divide $x^3 + 2x - 5$ by $x - 3$. We write

$$
\begin{array}{llll}
\textit{Dividend:} & 1 + 0 + 2 - 5 & \underline{|1 - 3} & \textit{Divisor} \\
& \underline{1 - 3} & |\underline{1 + 3 + 11} & \textit{Quotient} \\
& \quad + 3 + 2 & & \\
& \quad \underline{+ 3 - 9} & & \\
& \qquad\quad 11 - \ \ 5 & & \\
& \qquad\quad \underline{11 - 33} & & \\
& \qquad\qquad\quad 28 & \textit{Remainder} &
\end{array}
$$

Whence, $$\frac{x^3 + 2x - 5}{x - 3} = x^2 + 3x + 11 + \frac{28}{x - 3}.$$

Note that this becomes even more simple if we *write none of our numbers twice*. Thus, we may write

$$
\begin{array}{lll}
\textit{Dividend:} & 1 + 0 + \ \ 2 - \ \ 5 \ \underline{|1 - 3} & \textit{Divisor} \\
& \quad\ \ \underline{- 3 - \ \ 9 - 33} & \\
\textit{Quotient:} & 1 + 3 + 11 | + 28 & \textit{Remainder}
\end{array}
$$

which gives all the information in shorter time. This shorter form may be used only when the divisor is of the form $x \pm a$.

EXERCISES I: NOTE I — DETACHED COEFFICIENTS

Perform the indicated operations:

1. $(x^3 + 2x^2 - 5x + 3) \times (x + 2)$.
2. $(5x^3 - 3x^2 + 7) \times (3x^2 + x - 5)$.
3. $(x + 2x^2 + 3x^3 + x^4) \times (x - 2x^2 + 5x^3)$.
4. $(x + 2 + 2x^{-2} - 3x^{-2}) \times (x + 3 - x^{-1})$.
5. $(4x^3 + 2x - 1) \times (3x^2 + 2x)$.
6. $(3x^7 + 2x^6 - 5x + 4) \times (3x^2 - 7)$.
7. $(2x^3 - 3x + 7) \times (x^2 - 2x + 1)$.
8. $(x^3 + 6x^2 + 12x + 8) \div (x + 2)$.
9. $(6x^2 + 8x + 16) \div (x + 4)$.
10. $(8x^3 - 36x^2 + 54x - 27) \div (2x - 3)$.
11. $(12x^2 - 14x + 16) \div (6x + 5)$.
12. $(14x^5 - 13x^4 + 27x^2 - 5) \div (2x^2 + 3x - 7)$.
13. $(25x^{-4} + 5x^{-3} - 2x^{-2} + x^{-1} + 5) \div (5x^{-2} + 3x^{-1} + 2)$.
14. $(3x^3 + 2x^2 - 5x + 7) \div (3x - 1)$.
15. $(x^3 + x^2 - 4x + 6) \div (x + 3)$.
16. $(x^4 - 4x^2 - 5x + 10) \div (x - 2)$.
17. $[(x^4 + x^3 - x - 1) \div (x - 1)] \div (x + 1)$.
18. $(x^3 + 7x^2 + 15x + 25) \div (x + 5)$.
19. $(x^4 - x^3 - 6x^2 + x - 3) \div (x - 3)$.
20. $[(x^2 + 2x + 1) \times (x + 3)] \div (x + 1)$.
21. $[(2x^3 + 5x^2 + 12x + 5) \div (2x + 1)] \div (3x + 5)$.
22. $[(2x^3 + 4x^2 + 3)(8x^2 + 16x + 16)] \div [(4x + 4)(x + 3)]$.
23. $[(3x^3 + 4x + 7)(3x^2 - 5)] \div [(x + 5)(9x^2 + 2x + 3)]$.
24. $[x^{-5} + 4x^{-4} + 3x^{-3} - 2x^{-2} + x^{-1} + 5 + 3x] \times (x^{-1} + 3)$.

NOTE II. REMAINDER THEOREM; FACTORING

5. Factor Theorem. If a polynomial

$$P = Ax^n + Bx^{n-1} + \cdots + Lx + N$$

has a factor of the form $x - a$, we have $P = (x - a) \cdot Q$. Hence, the equation $P = 0$ is equivalent to the equation $(x - a)Q = 0$; since $x = a$ is a solution of this equation (as is seen by actually substituting a for x), we may say:

I. *If P has a factor $(x - a)$, then a is a root of the equation $P = 0$.*

Conversely, suppose a is a root of the equation $P = 0$. Dividing P by $x - a$, we should get some quotient Q and some remainder R, *where R is a number independent of x*:

$$P = Q(x - a) + R.$$

Now set $x = a$. Since $x = a$ is a solution of the equation $P = 0$, we have first $P = 0$ when $x = a$. Next, $(x - a)$ is zero when $x = a$. Hence, the above equation reduces to

$$0 = 0 + R,$$

that is, $R = 0$. In other words, the remainder obtained by dividing P by $x - a$ is zero, or P is exactly divisible by $x - a$.

II. *If $x = a$ is a root of $P = 0$, then P is divisible by $x - a$.*

6. Factors of $x^n \pm y^n$. These facts enable us to *factor* in many cases. For example, $x^n - 1 = 0$ is satisfied by setting $x = 1$. Likewise, $x^3 + 1, x^5 + 1$, etc., or, in general, $x^n + 1$ *where n is odd*, is divisible by $x + 1$, for the equation $x^n + 1 = 0$ (n odd) is satisfied by setting $x = -1$.

Finally, $x^n - 1$ is divisible by $x + 1$, if n is even, for $x^n - 1 = 0$ is satisfied by $x = -1$.

In like manner,

(1) $x^n - y^n$ *is* **always** *divisible by* $x - y$.

(2) $x^n - y^n$ *is divisible by* $x + y$ **if** n **is even.**

(3) $x^n + y^n$ *is divisible by* $x + y$ **if** n **is odd.**

Many forms may be factored upon this basis. Thus, $x^3 - y^3$ is divisible by $x - y$. By actual division, we find

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2).$$

Check:

$$\begin{array}{rcr}
1 + 1 + 1 & \quad \textit{i.e.} & x^2 + xy + y^2 \\
1 - 1 & \quad \textit{i.e.} & x - y \\
\hline
1 + 1 + 1 & & \\
-1 - 1 - 1 & & \text{multiplied} \\
\hline
1 + 0 + 0 - 1 & & x_3 \qquad - y^3
\end{array}$$

We find also, $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$;

$$x^4 - y^4 = [(x^2)^2 - (y^2)^2] = (x^2 - y^2)(x^2 + y^2)$$
$$= (x - y)(x + y)(x^2 + y^2);$$
$$x^5 - y^5 = (x - y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4);$$
$$x^5 + y^5 = (x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4);$$
$$x^6 - y^6 = [(x^3)^2 - (y^3)^2] = (x^3 - y^3)(x^3 + y^3)$$
$$= (x - y)(x^2 + xy + y^2)(x + y)(x^2 - xy + y^2);$$

or, also,
$$x^6 - y^6 = [(x^2)^3 - (y^2)^3] = (x^2 - y^2)\ (x^4 + x^2y^2 + y^4)$$
$$= (x - y)(x + y)(x^4 + x^2y^2 + y^4);$$

whence, comparing with the preceding, we find:

$$x^4 + x^2y^2 + y^4 = (x^2 + xy + y^2)(x^2 - xy + y^2),$$
$$x^7 \pm y^7 = (x \pm y)(x^6 \mp x^5y + x^4y^2 \mp x^3y^3 + x^2y^4 \mp xy^5 + y^6)$$

and so on.

These forms may be used as **type forms**, and other expressions may be factored by comparison with them, as in Chapter IV, p. 91. See also Chapter XIII, p. 331, where the same results are found by a different method.

EXERCISES I: NOTE II—FACTOR THEOREM

Factor the following expressions:

1. x^4-1.
2. x^5+1.
3. $8x^3-1$.
4. x^6-64.
5. x^4-16.
6. $64x^2-16$.
7. x^3-27.
8. x^5-32.
9. x^9-y^9.
10. x^3-a^3.
11. x^3+a^3.
12. $x^{2n}-y^{2n}$.
13. $8(x+y)^3+(x-y)^3$.
14. $(x+y)^3-(x-y)^3$.
15. $(x-y)^2-(x+y)^2$.
16. $(x+1)^3+1$.
17. $27r^3s^6-8$.
18. $(x+y)^5-(x-y)^5$.
19. $81p^4-625q^4$.
20. $m^5n^{10}-u^5v^{10}$.

7. Factors of Polynomials. Some polynomials may be factored by the above principles.

Ex. 1. To factor $P = x^4 - 4x^3 - x^2 + 16x - 12$, we may try the factors of the last term -12, since -12 must be the product of the constant terms in all the factors. Try ± 1, ± 2, ± 3, ± 4, ± 6, ± 12. Thus, to try $+1$ we set $x = +1$ in P; this gives $P =$ (for $x = 1$) $1 - 4 - 1 + 16 - 12 = 0$; hence, $x - 1$ is a factor of P. If we try all the other numbers above, we shall find that the factors of P are $(x-1)$, $(x-2)$, $(x+2)$, $(x+3)$. Check this by multiplying these factors together.

EXERCISES II: NOTE II — FACTORS OF POLYNOMIALS

Find by the factor theorem a factor of the left side of each of the following equations and by so doing find one real root of each equation:

1. $4x^3-5x+1=0$.
2. $x^3-x^2-x-15=0$.
3. $4x^3+16x^2-9x-63=0$.
4. $x^3-2x^2-x+2=0$.
5. $x^4+3x^2-6x+3=0$.
6. $x^4-10x^2-20x-16=0$.
7. $x^3-4x^2-16x+15=0$.
8. $x^3-8x^2+12x-5=0$.
9. $x^3-x-3x^2+3=0$.
10. $x^4+x-5x^3-5=0$.

8. Remainder Theorem. The work in § 5 proves another interesting fact. For we had

$$P = Q(x - a) + R,$$

where Q stands for the quotient and R for the remainder upon dividing P by $x - a$. Suppose $P_{(for\ x=a)} \neq 0$. Then substituting $x = a$, we get $P_{(for\ x=a)} = R$, since the term $Q(x - a)$ certainly falls out when $x = a$. In other words:

III. *The remainder found upon dividing P by $x - a$ is equal to the value of P when x is replaced by a.*

In long expressions it is easier to divide and find R than to substitute a in the place of x in P.

Thus, to find the value of $P = x^3 - 5x^2 + 7x + 3$, when $x = 4$, we may write

$$P_{(x=4)} = 4^3 - 5(4^2) + 7 \cdot 4 + 3$$

and actually compute 4^3, $5(4^2)$, etc.; or we may divide P by $x - 4$ and find R:

$$\begin{array}{rrrr|l}
x^3 & -\ 5x^2 & +\ 7x & +\ \ 3 & x - 4 \\
\cline{5-5}
x^3 & -\ 4x^2 & & & x^2 - x + 3 \\
\cline{1-2}
& -\ \ x^2 & +\ 7x & & \\
& -\ \ x^2 & +\ 4x & & \\
\cline{2-3}
& & 3x & +\ \ 3 & \\
& & 3x & -\ 12 & \\
\cline{3-4}
& & & 15 = R &
\end{array}$$

Then, $P_{(x=4)} = (4)^3 - 5(4^2) + 7 \cdot 4 + 3 = 15$, which the student may verify by actually computing $P_{(x=4)}$.

The work is even shorter in detached coefficients, see p. 355.

EXERCISES III: NOTE II — REMAINDER THEOREM

Find by the remainder theorem the value as shown in each of the following examples. Verify the first three by actually substituting the indicated value for the unknown.

1. $P = x^3 - 5x^2 + 15x - 75$, for $x = 5$.
2. $P = x^5 - 27x^4 + 15x^2 - 35x + 125$, for $x = -3$.
3. $P = x^2 - 35x + 17$, for $x = 2$.
4. $P = x^5 - 15x^4 + 25x^3 - 125x^2 + 50$, for $x = -4$.
5. $P = x^3 - 3x^2 + 15x - 20$, for $x = 3$.
6. $P = x^4 + 4x^3 - 12x^2 + 1 = 0$, for $x = -2$.

NOTE III. CHOICE AND CHANCE; PERMUTATIONS AND COMBINATIONS

9. Choice. The arrangements of objects and the possibilities of choice form the basis of this note.

As a typical example, suppose that I wish to select a route from Chicago to Liverpool, via New York, from among four railroads from Chicago to New York which I would consider, and three lines of steamers from New York to Liverpool. Having taken any railroad to New York, I may go to Liverpool on three different lines. Since I have four railroads from which to choose, the total number of arrangements is 4×3, or 12.

In general, if one choice is made in n ways and then another independent choice made in m ways, the total number of arrangements for the two choices is $n \times m$, or nm.

Likewise, for any series of choices, the total arrangements of all choices is the *product* of the separate numbers for the separate choices.

10. Chance. The *chance* of selecting a given object from among a number of objects decreases as the number of objects increases. If among five balls in a bag there is one white one, the chance of selecting the white one at a random choice is 1 to 5, or $\frac{1}{5}$. *In general, the chance of selecting one special object among n objects is $\frac{1}{n}$.*

The chance increases if there are more favorable possibilities. If there are *two* white balls among five, the chance of drawing a white ball is twice as great as it is if there is only one, *i.e.* the chance is $\frac{2}{5}$.

In general, the chance of selecting one of m objects out of a total of n objects is $\frac{m}{n}$.

EXERCISES I: NOTE III — CHOICE AND CHANCE

1. Two doors in one room and three doors in another (not adjoining) room open on to a common court. In how many ways may one go from one room to the other?

2. A man has three coats, two vests, and five pairs of trousers. In how many ways can he dress?

3. A boy can go to school by five different roads. In how many ways may he go and return? In how many ways can he arrange his trips on six different days?

4. There are eleven horses in a pasture and nine saddles in a barn. How many choices of saddle and horse may be made?

5. What is the chance of selecting a black ball out of a bag containing five black balls and one white ball?

6. A bag contains six grains of white corn and five grains of yellow corn. What is the chance of selecting a white grain the first time a grain is taken? If a white grain is drawn the first time and kept out, what is the chance of getting another white grain on the second drawing? What is the total chance that a white grain will be drawn both times?

7. What is the chance of throwing "heads" in tossing a coin?

8. If three coins are thrown up together, what is the chance that at least one will fall "heads"? What is the chance that two will fall "heads"?

9. Dice are usually cubes marked on the six faces with the numbers from 1 to 6. What is the chance of throwing a "2" with one die? What is the chance of throwing a "2" with two dice? What is the chance of throwing a double "2" (*i.e.* a "2" on each die) with two dice? What is the chance of throwing two numbers whose sum is ten with two dice?

11. Permutations. The number of arrangements of a given set of objects in order is called the number of **permutations** of them.

Thus, given five pictures to be hung upon a wall, any one may be placed at the extreme right, then any of the four remaining ones next, then any one of the three remaining ones next, then any one of the two remaining ones next, and finally the last one must be hung at the extreme left. The number of possibilities is given by § 9. There are five distinct choices; the first with 5 objects from which to choose, the second with 4, and so on; hence, the total number of possible arrangements is $5 \times 4 \times 3 \times 2 \times 1 = 120$.

In general, if n objects are to be arranged in an order, any one of the n objects may be put first, then any of the remaining $n-1$ next, then any of the remaining $n-2$ next, and so on to the last. The total number of possible arrangements is $n(n-1)(n-2)\cdots 4 \cdot 3 \cdot 2 \cdot 1$.

12. Permutations among a Limited Number. If less than the whole number of objects are to be arranged in order, the number of possible arrangements is evidently reduced.

Thus, if we desire to select four out of ten candidates for an office and arrange them in order of merit, the number of possible arrangements is smaller than if all ten were to be arranged in order of merit. Any of the ten may take first rank, any of the nine remaining second rank, any of the eight remaining third rank, any of the seven remaining fourth rank; but here we must stop. The total number of arrangements (or permutations) is

$$10 \times 9 \times 8 \times 7 = 5040;$$

whereas, if all ten were to be arranged in order, the total number of arrangements (or permutations) would be

$$10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 3{,}628{,}800.$$

In general, if from among n objects we are to make an arrangement of m objects in an order, we may choose any one first, then any of the remaining $n-1$ second, and so on for m steps. The last choice will be $n-m+1$ objects,

since $n-m$ will still remain after the last choice. The number of arrangements is, therefore,

$$n(n-1)(n-2)\cdots(n-m+2)(n-m+1).$$

13. Factorial Notation. A convenient notation for the kind of expressions just found consists in writing $2!$ for $2\cdot 1$; $3!$ for $3\cdot 2\cdot 1$; $4!$ for $4\cdot 3\cdot 2\cdot 1$, etc.; in general,

$$n! = n(n-1)(n-2)\cdots 4\cdot 3\cdot 2\cdot 1.$$

The sign $n!$ is read "**factorial *n*.**"

The permutations of n objects, m at a time, is often denoted by $P_{n,m}$. If $m=n$, all the objects are to be arranged; we then write $P_{n,n}$ for the number of permutations.

Using this notation, the results found above are

$$P_{n,n} = n! \text{ and } P_{n,m} = \frac{n!}{(n-m)!}.$$

EXERCISES II: NOTE III — FACTORIALS; PERMUTATIONS

Compute the value of:

1. $5!$ **2.** $6!$ **3.** $12!$ **4.** $(8!)\div(4!)$. **5.** $(5!\times 4!)\div(6!)$.

6. $P_{3,1}$; $P_{5,2}$; $P_{7,4}$; $P_{8,8}$; $P_{15,5}$; $P_{9,4}$; $P_{n+1,n-1}$; $P_{n+k,k}$; $P_{12,10}$.

Write in abbreviated form and compute the number of permutations of:

7. 8 objects taken 3 at a time.

8. 15 objects taken 7 at atime.

9. k objects taken 1 at a time.

10. $n+1$ objects taken $n-1$ at a time.

14. Combinations. If we merely wish to select objects from among a given set of objects *without arranging them in order*, many of the permutations become equivalent.

Thus, if among the ten candidates of the problem in § 12 we wish to select four *without placing them in order of merit*, the same four men would be arranged in $4! = 4\cdot 3\cdot 2\cdot 1 = 24$ different arrangements in § 12; these arrangements are all equivalent if we do not specify the

order. Hence, the number of *combinations* (*i.e.* not counting different arrangements of the same ones) is $\frac{1}{24}$ of the previous number, *i.e.*

$$\frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} = 210.$$

In general, if $C_{n,m}$ represents the number of **combinations** (*i.e.* selections not counting different arrangements of the same objects) among n objects chosen m at a time,

$$C_{n,m} = \frac{P_{n,m}}{P_{m,m}} = \frac{\frac{n!}{(n-m)!}}{m!} = \frac{n!}{(n-m)!\,m!}.$$

EXERCISES III: NOTE III — COMBINATIONS

1. Find the value of $(7!) \div (4! \times 3!)$.

2. Find the value of $C_{7,3}$; $C_{5,4}$; $C_{5,5}$.

3. Find the values of $C_{4,2}$; $P_{4,2}$; find $C_{4,2} \div P_{4,2}$.

4. Find $P_{5,5}$; $P_{8,5}$; hence, find $C_{8,5}$.

5. If a farmer has twelve horses, how many different teams of two horses each may he use?

6. How many different and distinct committees of five may be selected from a group of 15 men?

7. In how many ways may 3 books be selected from 12 books?

8. In how many ways may the sum seven be thrown with two dice marked on the faces with the numbers from 1 to 6? With three dice?

9. In how many ways may three debaters be chosen from a squad of 18? How many if six must be chosen?

10. In a plane are ten points, no three of which are in a straight line. How many triangles may be formed with three of the points as vertices?

11. In how many ways can four men and three women be selected from eight men and seven women?

NOTE IV. INEQUALITIES

15. Operations on Inequalities. We have used a few inequalities, but we have not worked with them systematically. The signs $<$ (read "less than") and $>$ (read "greater than") are already known. A few statements that will be understood at once are now given:

(1) If $a > b$, *then* $b < a$.

(2) If $a > b$ *and* $b > c$, *then* $a > c$.

(3) If $a \geqq b$ and $b \geqq c$, *then* $a \geqq c$. ($\geqq$ is read "greater than or equal to," and $\leqq$ is read "less than or equal to.")

(4) If $a \leqq b$ *and* $b \leqq c$, *then* $a \leqq c$.

(5) If $a > b$, *then* $ka > kb$ *if* $k > 0$.

(6) If $a > b$, *then* $ka < kb$ *if* $k < 0$, for *changing the sign evidently reverses the inequality.*

(7) If $a > b$ *and* $c > d$, *then* $a + c > b + d$.

(8) If $a > b$, *then* $a \pm x > b \pm x$ *where* x *is any number.*

These rules, together with a clear understanding of what is intended, enable us to work with all ordinary inequalities.

Ex. 1. For what values of x is $2x - 3 > 6 - x$?

Subtracting $6 - x$ from each side, we get

$$(2x - 3) - (6 - x) > 0,$$

or, $$3x - 9 > 0.$$

Add 9 to each side: $$3x > 9.$$

Divide both sides by 3: $$x > 3.$$

Hence, if $x > 3$, then $$2x - 3 > 6 - x.$$

16. Graphical Solutions. The problem just solved may be done graphically. Thus, let $l = 2x - 3$, $r = 6 - x$ where r and l denote the right and left sides of the above inequality. Plotting each of these on squared paper, we have two straight lines, as shown. It is clear from such a figure that $l > r$ whenever the line $l = 2x - 3$ is above the line $r = 6 - x$. This happens evidently for all values of x greater than $x = 3$, *for the lines cross at* $x = 3$, and they surely do not cross again.

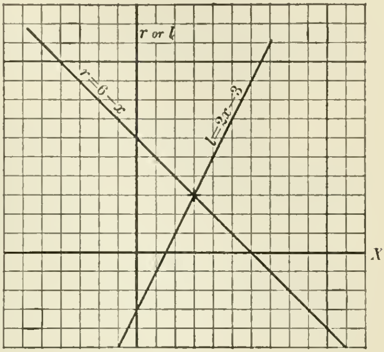


FIG. 72.

We may use this fact to advantage in harder examples. Let us find the values of x for which $x^2 + 2x - 8 > 0$. Call $l = x^2 + 2x - 8$ the left side and draw the figure; it is as shown (compare pages 183, 204). We see that $l > 0$ when the curve is above the main horizontal; this happens twice, once to the left of $x = -4$, once to the right of $x = +2$. These points are to be found by solving the equation $l = 0$, *i.e.*

$$x^2 + 2x - 0,$$

of which the solutions are

$$x = +2, \ x = -4.$$

Likewise,

$$l < 0, \textit{ i.e. } x^2 + 2x - 8 < 0$$

for all values of x *between* $x = -4$ and $x = +2$.

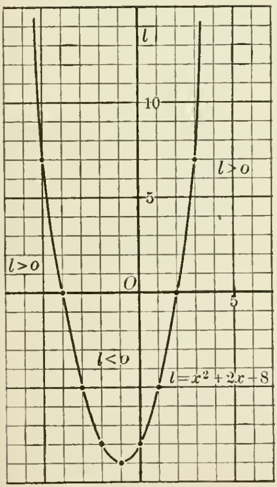


FIG. 73.

Similarly, in any case we may represent the left and right sides graphically and see when one exceeds the other. It is evident from a figure that we should first find when the two sides are *equal*, for the points for which this is true are points of intersection of the graphs.

EXERCISES I: NOTE IV—INEQUALITIES

Write down in the following examples the conclusions you could draw from what is given, as directed:

1. $10 > 6$; multiply both sides by 5; by -5; by $\frac{1}{2}$; by $-\frac{1}{2}$; divide by 2; divide by $\frac{1}{3}$; add 4 to each side; add -4; add -20; subtract 4; subtract 6. (Perform each operation on the *given* inequality only.)

2. $-10 < -6$; multiply by $+5$; by -5; by $\frac{1}{2}$; by $-\frac{1}{2}$; add 10 to each side; subtract 4.

3. $3 > 2$; multiply by x (if x is positive); divide by x (positive); add x to each side; subtract x.

4. $3 > 2$; multiply by x (if $x < 0$); add x; subtract x.

5. Given $3x - 2 > x + 4$, subtract x from each side; then add 2 to each side of the resulting inequality; then divide both sides by 2.

Draw the figures and find the values of x for which:

6. $3x - 2 > x + 4$. **9.** $x^2 > 6 - x$. **12.** $x^2 + 4x - 5 > 0$.

7. $x - 2 > 3 - x$. **10.** $x^2 > 5x - 4$. **13.** $x^2 - 2x - 3 > 0$.

8. $4x - 2 > 2x + 3$. **11.** $x^3 < x^2$. **14.** $x^3 + 2x^2 - 3 > 0$.

NOTE V. THE BINOMIAL THEOREM

17. Formula. We have learned how to write down certain powers of binomials by inspection. (See §§ 57, 62, pp. 93, 99.) Thus, we had

(1) $(a+b)^2 = a^2 + 2ab + b^2$ (p. 93),

(2) $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ (p. 99),

(3) $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ (p. 99),

and so on. We shall now prove the following formula, called the **binomial theorem**, which gives the result for any positive integral value of the exponent n:

$$(4)\quad (a+b)^n = \overset{(1st\ term)}{a^n} + \overset{(2d\ term)}{na^{n-1}b} + \overset{(3d\ term)}{\frac{n(n-1)}{2\,!}a^{n-2}b^2}$$

$$+ \overset{(4th\ term)}{\frac{n(n-1)(n-2)}{3\,!}a^{n-3}b^3} + \left\{\begin{matrix}\textit{student write}\\ \textit{5th term}\end{matrix}\right\} + \cdots$$

$$+ \overset{(rth\ term)}{\frac{n(n-1)(n-2)\cdots(n-r+2)}{(r-1)!}a^{n-r+1}b^{r-1}}$$

$$+ \overset{((r+1)th\ term)}{\frac{n(n-1)(n-2)\cdots(n-r+1)}{r\,!}a^{n-r}b^r} + \cdots$$

$$+ \overset{(nth\ term)}{ab^{n-1}} + \overset{(last\ term)}{b^n},$$

where $r\,!$ means $1 \cdot 2 \cdot 3 \cdots (r-1) \cdot r$, as on p. 364.

To prove this formula, we shall show that if it holds for any positive integral value of n, it holds also for the next higher value of n. For this purpose, assume for a

moment that (4) is correct and multiply both sides by $(a+b)$; this gives

$$(5)\quad (a+b)^{n+1} = \overset{(a \times 1st\ term)}{[a^{n+1}]} + \overset{(b \times 1st\ term)}{[a^n b} + \overset{(a \times 2d\ term)}{na^n b]}$$

$$+ \overset{(b \times 2d\ term)}{\Big[na^{n-1}b^2} + \overset{(a \times 3d\ term)}{\frac{n(n-1)}{2\,!}a^{n-1}b^2\Big]} + \overset{(b \times 3d\ term)+(a \times 4th\ term)}{\left\{\begin{matrix}\textit{student write}\\ \textit{this down}\end{matrix}\right\}} + \cdots$$

$$+ \overset{(b \times rth\ term)}{\left[\frac{n(n-1)(n-2)\cdots(n-r+2)}{(r-1)\,!}a^{n-r+1}b^r\right.}$$

$$+ \overset{(a \times (r+1)th\ term)}{\left.\frac{n(n-1)(n-2)\cdots(n-r+1)}{r\,!}a^{n-r+1}b^r\right]} + \cdots$$

$$+ \overset{(b \times n\ term)}{[nab^n} + \overset{(a \times last\ term)}{ab^n]} + \overset{(b \times last\ term)}{[b^{n+1}]};$$

or, collecting the terms, we have

$$(a+b)^{n+1} = a^{n+1} + (n+1)a^n b + \frac{(n+1)n}{2\,!}a^{n-1}b^2 + \cdots$$

$$+ \frac{(n+1)n(n-1)(n-2)\cdots(n-r+2)}{r\,!}a^{n-r+1}b^r + \cdots$$

$$+ (n+1)ab^n + b^{n+1},$$

which is the same as (4) with $(n+1)$ put in place of n.

It follows that if formula (4) *holds for any positive integral exponent n, it holds also for the next higher integral exponent $n+1$*; for we have derived (5) on the assumption that (4) is correct. Now we know (4) holds if $n=2$, for if $n=2$, the formula reduces to (1), which we know is correct. Hence, the formula must also hold if $n=3$, by our argument just given. Since it holds for $n=3$, it must hold for $n=4$; since it holds for $n=4$, it must hold for $n=5$; etc. In fact, we may say that the formula

(4) *holds for any positive integer n whatever*, for we should eventually reach any given one. Thus the formula is proved.

The style of argument just used is called **mathematical induction**, and it is useful in many other proofs.

A proof by *mathematical induction* shows: (1) that a formula is true for at least one particular integral value of one of the letters; then, (2) that if this formula is true for a given integral value of that letter, it is true also for the *next higher* integral value; from (1) and (2) it follows that the law is true when the letter mentioned has any integral value greater than the value actually tested. This reasoning may also be modified.

18. Notes and Examples. In the formula just given it should be noted that:

(1) *There are $(n+1)$ terms in all; e.g.* there are 5 terms in $(a+b)^4$.

(2) *The exponent of a is reduced by one from each term to the next one; the exponent of b is increased by one.*

(3) *The coefficients are*

$$\begin{array}{llllllll}
\text{for } n=1: & 1, & 1; \\
\text{for } n=2: & 1, & 2, & 1; \\
\text{for } n=3: & 1, & 3, & 3, & 1; \\
\text{for } n=4: & 1, & 4, & 6, & 4, & 1; \\
\text{for } n=5: & 1, & 5, & 10, & 10, & 5, & 1; \\
\text{for } n=6: & 1, & 6, & 15, & 20, & 15, & 6, & 1; \quad \text{etc., etc.}
\end{array}$$

[Let the student extend this table. A rule is readily formed for obtaining any of these numbers, — by adding any two successive ones in the same row we get the number directly below the second one of the two added.]

Ex. 1. $$(a+b)^4 = a^4 + 4\,a^{4-1}b + \frac{4(4-1)}{1\cdot 2}a^{4-2}b^2 + \frac{4(4-1)(4-2)}{1\cdot 2\cdot 3}a^{4-3}b^3 + \frac{4(4-1)(4-2)(4-3)}{1\cdot 2\cdot 3\,.\,4}a^{4-4}b^4$$
$$= a^4 + 4\,a^3b + 6\,a^2b^2 + 4\,ab^3 + b^4.$$

This result agrees with (3) above. Also that, if we did not know where to stop, we should find the next term exactly *zero* if we *did* write it down; hence, there is no danger of writing too many terms.

Ex. 2. $(a+b)^7 = a^7 + 7\,a^{7-1}b + \dfrac{7(7-1)}{2}a^{7-2}b^2 + \cdots$ (student complete), $\quad = a^7 + 7\,a^6b + 21\,a^5b^2 + \cdots$ (student complete).

Ex. 3.
$$(2\,x+3\,y)^5 = (2\,x)^5 + 5(2\,x)^4(3\,y) + \frac{5\cdot 4}{1\cdot 2}(2\,x)^3(3\,y)^2$$
$$+\frac{5\cdot 4\cdot 3}{1\cdot 2\cdot 3}(2\,x)^2(3\,y)^3 + \frac{5\cdot 4\cdot 3\cdot 2}{1\cdot 2\cdot 3\cdot 4}(2\,x)(3\,y)^4 + (3\,y)^5$$
$$= 32\,x^5 + 240\,x^4y + 720\,x^3y^2 + 1080\,x^2y^3 + 810\,xy^4 + 243\,y^5.$$

Ex. 4. $(2\,x-3\,y)^6 = [2\,x + (-3\,y)]^6 = (2\,x)^6 + 6(2\,x)^5(-3\,y)$ $+\dfrac{6\cdot 5}{1\cdot 2}(2\,x)^4(-3\,y)^2 + \dfrac{6\cdot 5\cdot 4}{1\cdot 2\cdot 3}(2\,x)^3(-3\,y)^3 + \cdots$ (student complete), $\quad = 64\,x^6 - 576\,x^5y + 2160\,x^4y^2 - 4320\,x^3y^3 + \cdots.$

Ex. 5. The 6th term of $(a+b)^{12}$ is given by putting $n=12$ and $r=6$ in the rth term of (4):

$$\text{6th term of } (a+b)^{12} = \frac{\cancel{12}\cdot 11\cdot \cancel{10}\cdot 9\cdot 8}{1\cdot \cancel{2}\cdot \cancel{3}\cdot \cancel{4}\cdot \cancel{5}}a^7b^5 = 792\,a^7b^5.$$

EXERCISES: NOTE V — BINOMIAL THEOREM

Write out in full:

1. $(a+b)^6$.
2. $(3\,m+2\,n)^4$.
3. $(x-y)^5$.
4. $(2\,x+y)^7$.
5. $(2\,x^2-3\,y)^3$.
6. $(4\,u^2+v^3)^4$.
7. $(1+x)^8$.
8. $(mp-1)^5$.
9. $(rs-t^2)^4$.

Write out the first three terms of:

10. $(x-y)^{12}$.
11. $(4\,x^2+3\,y^3)^5$.
12. $(a+b)^{20}$.
13. $(uv+1)^9$.
14. $(1-x)^{10}$.
15. $(2+4\,a)^6$.

16. Write out the 6th term of $(a+b)^{10}$.

17. Write out the 5th term of $(1+x)^{14}$.

18. Write out the 12th term of $(2\,a-3\,b)^{20}$.

NOTE VI. EUCLIDIAN METHOD H.C.F. AND L.C.M.

19. Euclidian Method. In Chapter V, p. 118, we defined H.C.F. and L.C.M., and showed how to find them if the given expressions could be easily factored. The following method applies to all *polynomials*, no matter how difficult the factoring may be.

Ex. 1. Find the H.C.F. of

$$A = 3x^2 - 11x + 6 \text{ and } B = 12x^3 + x^2 - 12x + 4.$$

Divide the expression of higher degree (B) by the other (A):

$$\begin{array}{rrrr|l}
B = 12x^3 + & x^2 - & 12x + & 4 & 3x^2 - 11x + 6 = A \\ \cline{5-5}
12x^3 - & 44x^2 + & 24x & & 4x + 15 = \textit{Quotient} \\ \cline{1-4}
 & 45x^2 - & 36x + & 4 & \\
 & 45x^2 - & 165x + & 90 & \\ \cline{2-4}
 & & 129x - & 86 & \text{or } 43(3x - 2) = \textit{Remainder}
\end{array}$$

Calling the quotient Q and the remainder R, we have, as always,

$$B = Q \cdot A + R$$

i.e. $12x^3 + x^2 - 12x + 4 = (4x + 15)(3x^2 - 11x + 6) + 43(3x - 2)$. *Any factor of both A and B must also be a factor of the remainder R, for $R = B - QA$, so that a factor of both A and B is a factor of $B - QA$, which is nothing but R.* For example, a factor of $12x^3 + x^2 - 12x + 4$ and $3x^2 - 11x + 6$ must also be a factor of $43(3x - 2)$.

All common factors of B and A are also common factors of A and R, and vice versa. Hence, we may take instead of the given problem the simpler one: to find the common factors of A and R; *i.e.* of $3x^2 - 11x + 6$ and $43(3x - 2)$.

This new problem may be done by inspection, or if this is too difficult in any case, by repeating the process. In our example it is clear that $3x - 2$ is a common factor; it is also the only one, by the argument just used. Hence, the required H.C.F. is $3x - 2$.

Ex. 2. Find the H.C.F. of $A = x^4 + 4x^3 + 2x^2 + x - 2$ and $B = 2x^5 + 5x^4 - 7x^3 - 2x^2 - 5x + 7$.

Dividing as above, we find

$$\begin{array}{r|l}
B = 2x^5 + 5x^4 - \ \ 7x^3 - 2x^2 - 5x + 7 & x^4 + 4x^3 + 2x^2 + x - 2 = A \\ \cline{2-2}
\underline{2x^5 + 8x^4 + \ \ 4x^3 + 2x^2 - 4x \phantom{{}+7}} & 2x - 3 = \textit{Quotient} \\
-3x^4 - 11x^3 - 4x^2 - \ \ x + 7 & \\
\underline{-3x^4 - 12x^3 - 6x^2 - 3x + 6} & \\
x^3 + 2x^2 + 2x + 1 & = \textit{Remainder}
\end{array}$$

As above, we may now take, instead of the given problem, the new problem of finding the H. C. F. of the divisor and the remainder:

$$\begin{array}{r|l}
x^4 + 4x^3 + 2x^2 + \ \ x - 2 & x^3 + 2x^2 + 2x + 1 \\ \cline{2-2}
\underline{x^4 + 2x^3 + 2x^2 + \ \ x \phantom{{}-2}} & x + 2 \\
2x^3 \qquad\qquad\quad - 2 & \\
\underline{2x^3 + 4x^2 + 4x + 2} & \\
-4x^2 - 4x - 4 & = -4(x^2 + x + 1) = \textit{Remainder}
\end{array}$$

Again, take the divisor and the remainder and discard the factor -4, which clearly cannot be a common factor:

$$\begin{array}{r|l}
x^3 + 2x^2 + 2x + 1 & x^2 + x + 1 \\ \cline{2-2}
\underline{x^3 + \ \ x^2 + \ \ x \phantom{{}+1}} & x + 1 \\
x^2 + \ \ x + 1 & \\
\underline{x^2 + \ \ x + 1} & \\
0 & = \textit{Remainder}
\end{array}$$

Since this remainder is *zero*, $x^2 + x + 1$ is a factor of $x^3 + 2x^2 + 2x + 1$; hence, it is the H. C. F. of the given expressions.

At any stage in the process we may find the H. C. F. by inspection, as in example 1, if it is easy to do so. Otherwise we repeat the same process as often as is necessary until we find a remainder zero; the last divisor is then the required H. C. F.

Numerical factors may be inserted or taken out at any time, for it is easy to see by inspection whether or not such factors are common factors of the given expressions. Usually no account is taken of these purely numerical factors.

Any common factors that can be found by inspection

should be removed at once. Sometimes the H. C. F. can be found in this manner, as in Chapter V.

The L. C. M. of two polynomials can be found by § 74, p. 129, after finding their H. C. F.

A precisely similar process holds for numbers; but care should be taken not to insert or discard numerical factors in dealing with problems in numbers.

EXERCISES: NOTE VI — H. C. F. AND L. C. M.

[Solve by inspection as in Chapter V wherever possible; otherwise use the process explained above.]

Find the H. C. F. and the L. C. M. of:

1. 35, 75, 25, and 65.

2. 42, 105, 147, and 63.

3. 1884 and 2079.

4. 3718, 5269, and 12,168.

5. $35\,a^4b^2$, $84\,a^2b^4$, and $63\,a^3b^3$.

6. $x^2-4\,x-21$ and $x^2-5\,x-14$.

7. x^3-1, x^2-1, and $(x-1)^2$.

8. $x^3+3\,x^2+3\,x+2$ and $x^3-2\,x^2-x-6$.

Find the H. C. F. of:

9. $x^3+x^2+3\,x+10$ and $x^3+x^2-5\,x-6$.

10. x^5-x^4-x+1 and $5\,x^4-4\,x^3-1$.

11. $x^2+6\,x+9$ and x^3+x^2+x+21.

12. $x^4-2\,x^3+3\,x^2-4\,x+2$ and $x^5-2\,x^4+3\,x^3+x^2-8\,x+5$.

13. $4\,x^3-18\,x^2-3+19\,x$ and $19\,x^2-12\,x^3+2\,x^4+9-6\,x$.

14. x^4-x^3+x-1 and x^4+x^3-x-1.

15. $3\,x^4-4\,x^2-4$ and $3\,x^4-8\,x^2+4$.

16. $x^3 - 1$ and $x^4 + 2x^3 + 3x^2 + 2x + 1$.

17. $x^3 + 3x^2 + 7x + 5$, $x^4 - 2x^3 + 4x^2 + 2x - 5$, and $x^5 + 5x^3 + x^2 + 5$.

18. $x^3 - 5x + 4$, $x^4 - 2x^3 + 1$, and $x^6 + 4x^3 - 3x - 2$.

19. $\begin{cases} x^5 + 3x^4 + 4x^3 - 9x^2 - 5x + 6, \\ x^5 + 3x^4 + 7x^3 + 10x - 12. \end{cases}$

20. $\begin{cases} 4 + 7x + x^2 + x^5 - x^4, \\ 7x + 4 - 2x^5 - 3x^3 - 3x^4. \end{cases}$

21. $\begin{cases} x^2 - 3x^3 + x^4 + 3x - 2, \\ 2x - 9x^2 + 4x^3 + 3. \end{cases}$

22. $x^3 - 6x^2 + 11x - 6$, $x^3 - 9x^2 + 26x - 24$, and $x^3 - 8x^2 + 19x - 12$.

Find the L. C. M. of the following:

23. $9xy^2$, $6x^2y^3$, and $15y^2z^2$.

24. $x^3 - 1$ and $x^3 + 1$.

25. $4x^2 - 9y^2$ and $4x^2 - 12xy + 9y^2$.

26. $(a + 3)$, $(a^2 - 9)$, $3a + 15$, and $15a - 45$.

27. $2x^3 - x^2 + 2 - 3x$, $6x^2 + 4x^3 - 4 - 2x$, and $4x^3 - 5x + 2$.

28. $x^2 - 11x + 24$, $x^2 - 6x - 16$, and $x^2 - x - 6$.

29. $c^2 - (a + b)^2$, $b^2 - (a + c)^2$, and $a^2 - (b + c)^2$.

30. $(x^2 - 2x + 1)$, $(x^2 - 1)^2$, $x^3 - 1$.

31. $x^4 - x^3 + x - 1$ and $x^4 + x^3 - x - 1$.

32. $x^3 - 9x^2 + 26x - 24$ and $x^3 - 10x^2 + 31x - 30$.

33. $x^2 + 6x + 9$ and $x^3 + x^2 + x + 21$.

34. $(a + b)^2 - (c + d)^2$ and $(a + b)^3 - (c + d)^3$.

35. $x^4 + 11x^2 + 25x$ and $x^4 + 2x^3 + 11x^2 + 10x + 25$.

36. $x^3 + 3x^2 + 7x + 5$ and $x^5 + 5x^3 + x^2 + 5$.

37. $x^3 + y^3$, $x^3 - y^3$, $(x - y)^3$, and $x^3 + 3x^2y + 3xy^2 + y^3$.

38. $x^2 + 2xy + y^2$, $x^2 - 2xy + y^2$, $x^2 - y^2$, and $x^2 + y^2$.

NOTE VII. CUBE ROOT AND HIGHER ROOTS

20. Introduction. When cube roots and higher roots of numbers are needed in practical work, they can be found approximately most easily by the use of logarithms (see § 163), which computers almost always use.

Any root of any number can be found approximately by trial, as in § 97. This process may be very long.

An old method, analogous to that used in § 97 for square root, is given here chiefly for its historic interest.

21. Cube Roots of Numbers.

Ex. 1. Find the cube root of 91609.86.

Notice that $1^3 = 1$, $10^3 = 1000$, $100^3 = 1{,}000{,}000$, and so on. Also, $.1^3 = .001$, $.01^3 = .000{,}001$, etc. Mark off the number into periods of three figures each, in each direction from the decimal point, thus: 91,609.86. This assists in estimating the first figure of the root.

The process is based on the formula

$$(a + b)^3 = a^3 + 3\,a^2b + 3\,ab^2 + b^3 = a^3 + b(3\,a^2 + 3\,ab + b^2),$$

and consists in choosing convenient values of a and of b and verifying; and then in repeating the process, choosing as a new value for a the whole root already found, after the manner of § 97.

	91,609.86	40
$40^3 =$	64,000	5
$3 \times 40^2 = 4800$	27,609.86	45
	91,609.86	45
$45^3 =$	91,125	.07
$3 \times 45^2 = 6075$	484.86	45.07
	91,609.86	45.07
$45.07^3 =$	91,550.911843	
	58.948157	

We take for a the largest convenient number whose cube is contained in the given number, in this case $a = 40$. If b is the remaining part of the root, then $3\,a^2b + 3\,ab^2 + b^3$ is the remainder of the cube. Since b is small compared with a, the principal term of this is $3\,a^2b$.

Hence, $3a^2b$ is nearly equal to 27,609.86. That is, $3 \times 40^2 \times b =$ 27,609.86 nearly. Hence, $b = (27{,}609.86) \div (3 \times 40^2)$ nearly $= 5$ nearly; this is called the **trial divisor.** Hence, the root is a little over 45. Start again: $a = 45$, cube a, subtract a^3, determine $b = .07$, etc., repeating this process, each time taking as a the part of the root already found. Each time b is found by dividing the remainder by $3a^2$. The last two figures of the root are found by simple division.

This work can be tabulated in slightly different form, as follows:

		91,609.86	40
$40^3 =$		64,000	
$3 \times 40^2 =$	4800	27,609.86	5
$3 \times 40 \times 5 =$	600		45
$5^2 =$	25		
$5 \times$	$5425 =$	27,125	
$3 \times 45^2 =$	6075	484.86	.07
$3 \times 45 \times .07 =$	9.45		
$.07^2 =$	.0049		
$.07 \times$	$6084.4549 =$	425.911843	45.07
$3 \times 45.07^2 =$	6093.9147	58.948157	.0097
			45.0797 :
			Answer to four places.

EXERCISES I: NOTE VII — CUBE ROOTS OF NUMBERS

Find the cube root of:

1. 262,144. **3.** .001906624. **5.** 12.812904.

2. 69,426,531. **4.** 259,694.072. **6.** 64,144.108027.

Find the cube root of the following to four figures:

7. 25,473. **8.** 46.32. **9.** 3.4674. **10.** 65,463,257.0423.

22. Cube Roots of Polynomials. The cube root of a polynomial that is a perfect cube may be found in a similar way. The polynomial should always be arranged according to the ascending or descending powers of the same letter. The most important points to remember are (1) the trial divisor is 3 times the square of the part of the root already found; (2) the complete divisor is $3a^2 + 3ab + b^2$ where a is the part of the root previously found and b is the new term.

Ex. 1. Find the cube root of:

$$114\,x^4 - 171\,x^2 - 135\,x - 27 + 8\,x^6 + 55\,x^3 - 60\,x^5.$$

	$8\,x^6 - 60\,x^5 + 114\,x^4 + 55\,x^3 - 171\,x^2 - 135\,x - 27$	$2\,x^2$
	$8\,x^6$	
$12\,x^4$	$-60\,x^5 + 114\,x^4 + 55\,x^3 - 171\,x^2 - 135\,x - 27$	$-5\,x$
$-30\,x^3$	$-60\,x^5 + 150\,x^4 - 125\,x^3$	
$+25\,x^2$		
$12\,x^4 - 30\,x^3 + 25\,x^2$		-3
$12\,x^4 - 60\,x^3 + 75\,x^2$	$-36\,x^4 + 180\,x^3 - 171\,x^2 - 135\,x - 27$	
$-18\,x^2 + 45\,x$		
$+9$		
$12\,x^4 - 60\,x^3 + 57\,x^2 + 45\,x + 9$	$-36\,x^4 + 180\,x^3 - 171\,x^2 - 135\,x - 27$	

Answer: $2\,x^2 - 5x - 3$

The first term of the root is evidently $2\,x^2$; hence, the first trial divisor is $3(2\,x^2) = 12\,x^4$. Dividing the first term of the first remainder by this trial divisor, we find the next term of the root: $-5\,x$. The first divisor completed is, therefore, $12\,x^4 + \{3(2\,x^2)(-5\,x) + (-5\,x)^2\} = 12\,x^4 - 30\,x^3 + 25\,x^2$. The remaining steps are repetitions of these.

23. Higher Roots. Since the fourth power is the square of the square, the *fourth root* is the square root of the square root. Likewise, the *sixth root* is the cube root of the square root. Other roots may be found in a similar manner, or by a method similar to that used for cube root. Any root of any number can be found approximately by a figure (p. 190), or by logarithms (§ 165, p. 342).

EXERCISES II: NOTE VII — CUBE ROOTS OF POLYNOMIALS

Find the cube root of:

1. $343\,a^3 - 441\,a^2b + 189\,ab^2 - 27\,b^3.$

2. $8\,m^6 - 12\,m^5n + 30\,m^4n^2 - 25\,m^3n^3 + 30\,m^2n^4 - 12\,mn^5 + 8\,n^6.$

3. $k^6 + 12\,k^5 + 63\,k^4 + 184\,k^3 + 315\,k^2 + 300\,k + 125.$

4. $a^3 - 12\,a^2 + 54\,a - 112 + \dfrac{108}{a} - \dfrac{48}{a^2} + \dfrac{8}{a^3}.$

5. $6\,x^4 + \dfrac{6}{x^4} + \dfrac{1}{x^6} + 20 + \dfrac{15}{x^2} + x^6 + 15\,x^2.$

6. $184\,c^3 + 315\,c^2 + 63\,c^4 + 300\,c + 125 + 12\,c^5 + c.$

NOTE VIII. LIMITS AND INFINITE SERIES; IRRATIONAL NUMBERS

24. Introduction. In this article we shall state briefly the fundamental notions connected with limits. These propositions are of a far more intricate character than are most of the topics treated in this book, and it is recommended that these articles be studied only by mature students.

25. Limits. *In case the difference between a variable quantity* v *and a constant quantity* k *can be made to become and remain as small numerically as we please, the variable* v *is said to approach the constant* k *as its* **limit**, *and we write*

$$\text{Lim } v = k.$$

Usually we shall have some definite process for determining whether or not the difference between the variable and the constant can be made to become and remain as small as we please.

Ex. 1. Imagine any porous body (*e.g.* a brick) soaked in water till its weight is increased. Let k be the weight of the body when dry, and let v be the weight when wet. If the body is heated, the water passes off in steam, and the weight v is variable. It is evident that by continued heating we can drive off as much of the water as we please. Hence, we can make the difference $v - k$ become and remain as small as we please; we say:

$$\text{Lim } v = k.$$

Ex. 2. It is found by trial that a rubber balloon will burst when it is so filled with gas as to have a diameter of six feet. If v means the volume of the balloon, it is evident that we may

expand balloon so that its volume is *as near as we please* to $\frac{4\pi}{3}\cdot(6)^3$ cu. ft. $= 288\pi$ cu. ft.; hence,

$$\text{Lim } v = 288\pi.$$

In this example the variable v cannot be made *equal* to its limit, for the balloon would burst if expanded to exactly six feet diameter.

In example 1, on the contrary, the variable may *equal* its limit, for we may drive off all the water and leave $v = k$.

Ex. 3. Consider the expression:

$$v = \frac{1-(\frac{1}{2})^n}{1-\frac{1}{2}}.$$

We may make this expression become and remain as near to $\frac{1}{1-\frac{1}{2}}$ as we please, by making n sufficiently large.

For let $$k = \frac{1}{1-\frac{1}{2}} = 2.$$

Then, $$v - k = \frac{1-(\frac{1}{2})^n}{1-\frac{1}{2}} - \frac{1}{1-\frac{1}{2}} = -\frac{(\frac{1}{2})^n}{1-\frac{1}{2}} = -\left(\frac{1}{2}\right)^{n-1},$$

or, $$v - k = -\frac{1}{2^{n-1}} = -\frac{1}{2\cdot 2\cdots((n-1)\text{ times})\cdots 2}.$$

Since the denominator contains as many factors 2 as we please, this fraction can evidently be made to become and remain as small as we please, numerically, by taking n large enough. Hence,

$$\text{Lim}\left[\frac{1-(\frac{1}{2})^n}{1-\frac{1}{2}}\right] = 2,$$

where we understand that the limit is to be taken by making n as large as we please.

Ex. 4. Similarly,
$$v = \frac{1-\left(\frac{1}{x}\right)^n}{1-\frac{1}{x}},$$

where x is any number numerically greater than 1, approaches as its limit

$$\frac{1}{1-\frac{1}{x}} \text{ or } \frac{x}{x-1}.$$

For let
$$k = \frac{1}{1 - \frac{1}{x}} = \frac{x}{x-1};$$

then,
$$(v - k) = \frac{1 - \left(\frac{1}{x}\right)^n}{1 - \frac{1}{x}} - \frac{1}{1 - \frac{1}{x}} = - \frac{\left(\frac{1}{x}\right)^n}{1 - \frac{1}{x}} = - \frac{1}{(x-1)\, x^{n-1}},$$

or,
$$v - k = - \frac{1}{(x-1) \cdot x \cdot x \cdots ((n-1) \text{ times}) \cdots x}.$$

If x is numerically greater than 1, the denominator can be made as large numerically as we please; hence, $v - k$ can be made to become and remain as small numerically as we please by taking n sufficiently large. Hence,

$$\operatorname{Lim} \left[\frac{1 - \left(\frac{1}{x}\right)^n}{1 - \frac{1}{x}} \right] = \frac{1}{1 - \frac{1}{x}} = \frac{x}{x-1},$$

if x is numerically greater than 1.

26. Infinite Series. The last examples above have a direct application in the question of infinite series. By an **infinite series** we mean an unending sequence of terms connected by $+$ signs:

$$a_0 + a_1 + a_2 + a_3 + \cdots + a_n + \cdots.$$

(It is improper to say that an infinite series is the sum of an unending sequence of terms, for the word "sum" is defined only for a finite number of terms.)

We cannot find the sum of such an infinite series directly, but we *can* add together as many of the terms as we please; thus, we can find *the sum of the first n terms:*

$$S_n = a_0 + a_1 + a_2 + \cdots + a_{n-1}$$

by direct addition. We say that the *sum of the series* is the *limit* of S_n as n increases indefinitely, if there *is* a limit:

$$S = \operatorname{Lim} S_n, \quad \text{if } \operatorname{Lim} S_n \text{ exists.}$$

27. Infinite Geometric Series. As an example, consider an infinite geometric series (see § 159, p. 328):

$$a + ar + ar^2 + ar^3 + \cdots + ar^n + \cdots,$$

where r is numerically less than 1. The sum of the first n terms (see p. 329) is

$$S_n = a + ar + ar^3 + \cdots + ar^{n-1} = a \cdot \frac{1 - r^n}{1 - r}.$$

The sum of the *infinite series* is therefore

$$S = \text{Lim}\,(S_n) = \text{Lim}\left(a \cdot \frac{1 - r^n}{1 - r}\right).$$

Since r is numerically less than 1, let us put $r = \dfrac{1}{x}$; then, x is numerically greater than 1, and we have

$$S = \text{Lim}\left(a \cdot \frac{1 - r^n}{1 - r}\right) = \text{Lim}\left[a \cdot \frac{1 - \left(\frac{1}{x}\right)^n}{1 - \frac{1}{x}}\right] = a\frac{1}{1 - \frac{1}{x}},$$

or,

$$S = a\frac{1}{1 - r} = \frac{a}{1 - r},$$ by example 4, § 25.

Hence, the sum of an infinite geometric series in which r is numerically less than 1, *is* $\dfrac{a}{1 - r}$.

Ex. 1. In the infinite series

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \cdots$$

$$a = 1,\ r = \tfrac{1}{2};\ \text{hence,}\ S = \frac{a}{1 - r} = \frac{1}{1 - \frac{1}{2}} = 2.$$

Ex. 2. The series

$$1 - \tfrac{1}{2} + \tfrac{1}{4} - \tfrac{1}{8} + \tfrac{1}{16} - \tfrac{1}{32} + \tfrac{1}{64} - \cdots$$

gives $\quad a = 1,\ r = -\tfrac{1}{2};$ hence, $S = \dfrac{a}{1 - r} = \dfrac{1}{1 - (-\frac{1}{2})} = \dfrac{2}{3}.$

Ex. 3. The series

$$3 + \tfrac{3}{5} + \tfrac{3}{25} + \tfrac{3}{125} + \tfrac{3}{625} + \cdots$$

gives $\quad a = 3,\ r = \frac{1}{5}$; hence, $S = \dfrac{a}{1-r} = \dfrac{3}{1-\frac{1}{5}} = \dfrac{15}{4} = 3\frac{3}{4}$.

Ex. 4. The series

$$3 + \tfrac{3}{10} + \tfrac{3}{100} + \tfrac{3}{1000} + \cdots$$

gives $\quad a = 3,\ r = \frac{1}{10}$; hence, $S = \dfrac{a}{1-r} = \dfrac{3}{1-\frac{1}{10}} = \dfrac{30}{9} = \dfrac{10}{3}$.

Note: this series may be written $3.3333 \cdots$ in the form of a *repeating decimal*, and we may write $\frac{10}{3} = 3.3333 \cdots$

Ex. 5. The repeating decimal $.27272727 \cdots$, which is often indicated by $.\dot{2}\dot{7}$, is equivalent to the geometric series:

$$\tfrac{27}{100} + \tfrac{27}{10000} + \tfrac{27}{1000000} + \cdots;$$

here $\quad a = \frac{27}{100},\ r = \frac{1}{100}$; hence, $S = \dfrac{a}{1-r} = \dfrac{\frac{27}{100}}{1-\frac{1}{100}} = \dfrac{27}{99} = \dfrac{3}{11}$; consequently, $\frac{3}{11} = .272727 \cdots$.

Ex. 6. The decimal $5.743216216216216 \cdots$ (often written $5.743\dot{2}1\dot{6}$) repeats the figures 216 forever; it is equivalent to the series

$$5.743 + \frac{216}{10^6} + \frac{216}{10^9} + \cdots.$$

Taking $a = \dfrac{216}{10^6},\ r = \dfrac{1}{10^3}$, we find

$$\frac{216}{10^6} + \frac{216}{10^9} + \cdots = \frac{\frac{216}{10^6}}{1 - \frac{1}{10^3}} = \frac{216}{999000} = \frac{24}{111000},$$

hence, $\quad 5.743216216216 = 5 + \dfrac{743}{1000} + \dfrac{24}{111000} = 5 + \dfrac{82497}{111000}$.

In general, any repeating decimal may be regarded as a terminating decimal + a certain geometric series; such a decimal may therefore be evaluated by the preceding formula for the sum of an infinite geometric series; the result is always a rational fraction.

EXERCISES I: NOTE VIII—INFINITE GEOMETRIC SERIES

Find the values of the following infinite geometric series:

1. $2+\frac{2}{5}+\frac{2}{5^2}+\frac{2}{5^3}+\cdots$.

2. $2-\frac{2}{5}+\frac{2}{5^2}-\frac{2}{5^3}+\cdots$.

3. $3+2+\frac{4}{3}+\frac{8}{9}+\frac{16}{27}+\cdots$.

4. $\frac{5}{7}+\frac{15}{35}+\frac{45}{175}+\frac{135}{875}+\cdots$.

5. $-2-\frac{3}{2}-\frac{9}{8}-\frac{27}{32}-\frac{81}{128}-\cdots$.

Find the values of the following repeating decimals; check each answer by long division:

6. $2.222\cdots$.
7. $.7777\cdots$.
8. $.232323\cdots$.
9. $5.133333\cdots$.
10. $42.716161616\cdots$.
11. $.0454545\cdots$.
12. $10.1010101010\cdots$.
13. $26.308308308\cdots$.
14. $83.83838383\cdots$.

28. Other Infinite Series. We have seen how to find the sum of an infinite geometric series if r is numerically less than 1.

In general, as in § 26, we say that the sum of any series

$$a_0+a_1+a_2+\cdots+a_n+\cdots \tag{1}$$

is
$$S=\operatorname{Lim} S_n=\operatorname{Lim}\,(a_0+a_1+a_2+\cdots+a_{n-1})$$

if this limit exists.

If $\operatorname{Lim} S_n$ *exists, the series* (1) is called **convergent**; otherwise (1) is called **divergent**.

Ex. 1. Thus, $1+2+4+8+\cdots$ is an infinite geometric series which *diverges*, for

$$S_1=1,\ S_2=3,\ S_3=7,\ S_4=15,\cdots,$$

$$S_n=(1+2+4+\cdots+2^{n-1}),$$

and S_n becomes greater than any quantity one may name; hence, $\operatorname{Lim} S_n$ does not exist, and the series diverges.

Ex. 2. Another divergent geometric series is

$$1 - 1 + 1 - 1 + 1 - 1 \cdots, \text{ where } a = 1 \text{ and } r = -1;$$

here $S_n = +1$ if n is even, and $S_n = 0$ if n is odd. There is therefore no constant which S_n approaches; hence, Lim (S_n) does not exist, and the series diverges.

Ex. 3. A series which is not geometric, and which diverges, is

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8} + \tfrac{1}{9} + \cdots.$$

This series diverges, for if we arrange it in the form

$$1 + \tfrac{1}{2} + (\tfrac{1}{3} + \tfrac{1}{4}) + (\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}) + \cdots,$$

inclosing the next eight terms ending with $\frac{1}{16}$, then 16 terms ending with $\frac{1}{32}$, etc., we see that each parenthesis exceeds $\frac{1}{2}$; hence,

$$S_n > 1 + \tfrac{1}{2} + \tfrac{1}{2} + \cdots \text{ (as many times as we please)} + \tfrac{1}{2}.$$

It follows that S_n approaches no limit; hence this series *diverges.*

EXERCISES II: NOTE VIII—OTHER INFINITE SERIES

1. Show that the series $1 + 3 + 5 + 7 + 9 + \cdots$ diverges.

2. Show that the series $2 - 1 + 3 - 1 + 4 - 1 + \cdots$ diverges.

3. Show that the series $1 + \frac{1}{2} - 1 + \frac{1}{4} + 1 + \frac{1}{8} - 1 + \frac{1}{16} + \cdots$ diverges.

4. Show that any series $A_0 + A_1 + A_2 + \cdots$ will diverge if A_n does not approach zero as n increases indefinitely.

5. Show that if $a_0 + a_1 + a_2 + \cdots$ is a convergent series of positive terms, and if $b_0 + b_1 + b_2 + \cdots$ is another series of positive terms such that $b_n < a_n$, then the second series converges also.

29. Irrational Numbers. We have already considered many irrational numbers; thus, $\sqrt{2}$ is irrational. (See pp. 184, 186.) In fact, if the positive integer a is not the perfect square of an integer, $\sqrt{a}$ is irrational, for the supposition $\sqrt{a} = \dfrac{m}{n}$, where m and n are each integers without a common factor, leads to the absurdity $a = \dfrac{m^2}{n^2}$, *i.e.* an integer is equal to a fraction.

A **rational number** is the quotient of two integers ; an **irrational number** is any real number that is not rational. (See pp. 186, 284.)

In case of $\sqrt{2}$ we saw that we could get a rational number whose square is as close to 2 as we please, and we wrote

$$(1.4)^2 < 2 < (1.5)^2,$$

$$(1.41)^2 < 2 < (1.42)^2,$$

$$(1.414)^2 < 2 < (1.415)^2.$$

In fact, the square of any rational number is *either greater than* 2 or *less than* 2. Thus, the rational numbers are divided into two classes according as their squares are greater than or less than 2. This division into two classes practically *defines* $\sqrt{2}$, as above.

Whenever, in such a fashion, all rational numbers are divided into two classes such that any number in one of the classes is greater than any number in the other class, we say that this division *defines a certain irrational number*, which is called the **cut number**. Thus, $\sqrt{2}$ is the cut number which separates the rational numbers whose squares are less than 2 from those whose squares are greater than 2. We may also think of $\sqrt{2}$, for example, as the limit of a sequence of rational numbers

$$a_0, a_1, a_2, \cdots, a_n, \cdots,$$

where $(2 - a^2_n)$ approaches zero as n increases indefinitely. Thus, $\sqrt{2}$ is the limit of the sequence of the numbers

$$1.4, 1.41, 1.414, 1.4142, \cdots,$$

which are obtained in the ordinary square root process.

30. Operations on Irrationals. Any irrational can be expressed by means of rationals to any degree of exactness

required. Thus, $\sqrt{2}$ may be expressed, by means of one of the numbers, 1.4, 1.41, 1.414, ⋯ to any desired number of decimal places. Similarly, for any irrational, we may decide upon two integers between which it lies, then tenths between these, and so on; eventually the irrational is expressed to any number of decimal places desired.

If two irrationals are given, say $\sqrt{2}$ and $\sqrt{3}$, we may express each of them to any number of decimal places. Having done so, we may add, subtract, multiply, or divide these approximate values of the two to get the sum, difference, product, or quotient of the approximate values.

We now *define* the *sum* of two irrationals, or of one rational and one irrational, to be the limit approached by the sum of the approximations as the number of true decimal places in each approximation increases indefinitely. Likewise, the *difference*, *product*, *quotient*, etc., are defined as the limits of the corresponding approximate values. It can be shown that by these definitions the axioms of § 24, p. 35, remain true.

As an example consider $\sqrt{2} \times \sqrt{3}$. The rationals that define $\sqrt{2}$ are all those whose squares are less than 2; the rationals that define $\sqrt{3}$ are all those whose squares are less than 3. If we multiply these approximate rational values together, it is clear that the approximate products are all those rational numbers whose squares are less than 6. It follows that $\sqrt{2} \cdot \sqrt{3} = \sqrt{6}$, a result previously obtained by analogy, without strictly logical proof. Similarly, $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$, and, in general, $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$. This is the rule of p. 286, which may now be strictly proved.

The operations on radical quantities and on other irrationals used above should finally be revised in the manner just shown in order to justify them completely; but it will not be advisable to do this in detail in this book. It should be noticed that the definition given on p. 183, by

means of the graph $y = x^2$, really includes the notion of approximation just used. In general, for elementary purposes, a geometrical definition of irrational numbers, by the graph, mentioned on p. 186, will be found most suitable, since it is easy to follow and is also quite in accordance with the most logical definitions.

EXERCISES III: NOTE VIII — OPERATIONS ON IRRATIONALS

1. Write out a definition of $\sqrt[3]{2}$ as a *cut* number.

2. Give five terms of a sequence of increasing numbers whose limit is $\sqrt{5}$.

3. Give five terms of a sequence of increasing numbers whose limit is $\sqrt{3}$; also a sequence of decreasing numbers whose limit is $\sqrt{3}$.

4. Multiply $\sqrt{3}$ by $\sqrt{5}$; show the result is a *cut* number which expresses the product of the *cut* numbers which give $\sqrt{3}$ and $\sqrt{5}$, respectively.

5. Add the increasing sequences of numbers which approach $\sqrt{3}$ and $\sqrt{5}$, respectively, term by term; define the sum $\sqrt{3} + \sqrt{5}$ by this means as a *cut* number.

6. Prove $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$. **7.** Prove $\sqrt{a} \div \sqrt{b} = \sqrt{\frac{a}{b}}$.

8. Show that the graphs of $y = x^2$ and $y = 2$ serve to define the cut number $\sqrt{2}$.

9. Show that the graphs $y = x^2$ and $x^2 + y = 1$ serve to define the cut numbers $x = \pm\sqrt{\frac{1}{2}}$.

10. Show that the graphs of $y = x^3$, $y = x^4$, $y = x^5$, serve to define all cube, fourth, fifth roots. (See p. 190.)

11. Show that the curve $x = 10^y$ (see p. 336) defines $\log x$ as a cut number for every value of x.

HINT. Show that 10^y can be really found for every rational y; hence $\log x$ can be found for values of x as near any given value of x, as we please.

NOTE IX. IMAGINARY AND COMPLEX NUMBERS

31. Introduction. In the work of this book we have used the following classes of numbers:

(1) Positive integers, defined by counting (see p. 1).

(2) Positive rational fractions, the quotients of pairs of integers (see pp. 1, 118, 186).

(3) Zero, defined by the equation $0 + a = a$ (see p. 15).

(4) Negative integers and fractions, defined by the equation $a + (-a) = 0$ (see p. 14).

(5) Irrational numbers, defined by a *cut* (p. 387). These five classes of numbers are called **real numbers.**

Just as mankind has found it desirable to invent and use these various kinds of numbers, it is convenient to invent and use a sixth kind of number.

With real numbers alone we may perform any additions, subtractions, multiplications, divisions, except division by zero. We may also raise any number to any integral power, and we may extract integral roots of positive numbers. Indeed, we may extract any *odd* root of a *negative* number; but an *even root of a negative number cannot be expressed* by the numbers we know at present.

For this reason, the new kind of numbers arise in solving even very simple quadratic equations. (See pp. 182, 212.) Thus, $x^2 + 1 = 0$ gives $x^2 = -1$ or $x = \pm\sqrt{-1}$, but $\sqrt{-1}$ is an even root of a negative number, and cannot be expressed in terms of the numbers of the five classes above.

32. Definitions. We shall denote the new number $\sqrt{-1}$ by the letter i:

(1) $$\sqrt{-1} = i, \quad i^2 = (\sqrt{-1})^2 = -1;$$

and we shall work with this new number, as far as possible, according to the rules for real numbers, observing always the axioms of § 24, p. 35. (See note, § 36.)

The product bi of any real number b and the new number i is called a **pure imaginary number.** The square of such a number is

(2) $(bi)^2 = (bi)(bi) = b \cdot i \cdot b \cdot i = bbii = b^2 i^2 = b^2(-1) = -b^2$

by IV, p. 35. Likewise, $(-bi)^2 = -b^2$. Hence, we say

(3) $\sqrt{-b^2} = \pm bi$, and $\sqrt{-x} = \sqrt{-(\pm\sqrt{x})^2} = \pm\sqrt{x} \cdot i$

if $x > 0$, so that *the square root of any negative number* $-x$ *may be written as the pure imaginary* $\pm\sqrt{x} \cdot i$. We shall always do this at once in any problem.*

The sum $(a + bi)$, where a and b are real numbers, is called a **complex number**, or simply an **imaginary number.**

33. Direct Operations. Addition, subtraction, and multiplication are performed by working as if i were an unknown letter, and replacing i^2 by -1 wherever it occurs.

Ex. 1. $(2 + 3i) + (4 + 5i) = 6 + 8i$,

and, in general, $(a + bi) + (c + di) = (a + c) + (b + d)i$.

Ex. 2.
$$\begin{aligned}(2 + 3i)(4 + 5i) &= 8 + 22i + 15i^2\\ &= 8 + 22i - 15 = -7 + 22i,\end{aligned}$$

and, in general,
$$\begin{aligned}(a + bi)(c + di) &= ac + (bc + ad)i + bdi^2\\ &= ac + (bc + ad)i - bd\\ &= (ac - bd) + (bc + ad)i.\end{aligned}$$

These operations employ only the direct results of the axioms (p. 35), and the statement $i^2 = -1$.

---

* There is not the same advantage in dealing with square roots of negative quantities and with square roots of imaginary numbers, in deciding which of the two answers shall be denoted by the sign $\sqrt{}$, as there was in dealing with square roots of positive quantities. In examples, however, we shall take $\sqrt{-x} = +\sqrt{x}\, i$, to avoid undesirable complication.

Ex. 3. $(2+3i)(2-3i)=2^2-(3i)^2=4-(-9)=13,$
and, in general,

$$(a+bi)(a-bi)=a^2-(bi)^2=a^2-(-b^2)=a^2+b^2.$$

The result in this problem does not contain the letter i; hence, it is a *real* number.

34. Conjugate Complex Numbers. The importance of Ex. 3 above leads us to emphasize it by calling $a-bi$ the **conjugate** to $a+bi$. *The product of two conjugate complex numbers* $a+bi$ and $a-bi$ *is* the *real positive number* a^2+b^2. This is often stated by saying that *the sum of two squares is the product of the imaginary factors* $a+bi$ *and* $a-bi$:

$$a^2+b^2=(a+bi)(a-bi). \tag{4}$$

35. Division. Division of complex numbers is based on this fact, for the divisor (or denominator) may be made real by multiplying both divisor and dividend (or both numerator and denominator) by the conjugate to the divisor (or denominator).

Ex. 1. $$\frac{-7+22i}{2+3i}=\frac{-7+22i}{2+3i}\cdot\frac{2-3i}{2-3i}=\frac{-14+65i-66i_2}{4-9i^2}$$
$$=\frac{-14+65i+66}{4+9}=\frac{52+65i}{13}=4+5i.$$

Check: $(2+3i)(4+5i)=-7+22i$. (See Ex. 2, § 33.)

Ex. 2. $$\frac{1}{i}=\frac{1}{i}\cdot\frac{-i}{-i}=\frac{-i}{-i^2}=\frac{-i}{+1}=-i.$$

(We might have multiplied numerator and denominator by $+i$ instead of $-i$; but $-i$ was taken in order to observe the general rule.)

Ex. 3. $$\frac{2}{1+i}=\frac{2}{1+i}\cdot\frac{1-i}{1-i}=\frac{2-2i}{2}=1-i.$$

Ex. 4. $$\frac{4-5i}{3-2i}=\frac{4-5i}{3-2i}\cdot\frac{3+2i}{3+2i}=\frac{12-7i-10i^2}{9-4i^2}$$
$$=\frac{12-7i+10}{9+4}=\frac{22-7i}{13}-\frac{23}{13}-\frac{7}{13}i.$$

36. Further Examples. If square roots of other negative numbers than -1 occur, we reduce them at once to the form above by the relation (3), *i.e.* $\sqrt{-x} = \pm\sqrt{x}i$.

In the following examples the sign $+$ is chosen in each case:

Ex. 1. $(2+\sqrt{-4})+(3+2\sqrt{-9})$

$$=(2+2\sqrt{-1})+(3+2\cdot 3\sqrt{-1})=5+8\,i.$$

Ex. 2. $(4-\sqrt{-9})\times(2+\sqrt{-16})=(4-3\,i)(2+4\,i)$

$$=20+10\,i.$$

Ex. 3. $\dfrac{1+\sqrt{-4}}{1-\sqrt{-4}}=\dfrac{1+2\,i}{1-2\,i}\cdot\dfrac{1+2\,i}{1+2\,i}=\dfrac{-3+4\,i}{5}=-\dfrac{3}{5}+\dfrac{4}{5}i.$

Ex. 4. $(2+\sqrt{-3})(1+\sqrt{-3})=(2+\sqrt{3}\,i)(1+\sqrt{3}\,i)$

$$=2+3\sqrt{3}\,i+3\,i^2=-1+3\sqrt{3}\,i.$$

Ex. 5. $\dfrac{4+\sqrt{-5}}{2+\sqrt{-3}}=\dfrac{4+\sqrt{5}\,i}{2+\sqrt{3}\,i}\cdot\dfrac{2-\sqrt{3}\,i}{2-\sqrt{3}\,i}$

$$=\frac{8+(2\sqrt{5}-4\sqrt{3})\,i-\sqrt{15}\,i^2}{4-3\,i^2}$$

$$=\frac{(8+\sqrt{15})+(2\sqrt{5}-4\sqrt{3})\,i}{7}$$

$$=\frac{8+\sqrt{15}}{7}+\frac{2\sqrt{5}-4\sqrt{3}}{7}i.$$

Failure to follow the directions given above may lead to error, for although some of the rules of ordinary algebra hold, there are other rules which do not hold (under the agreements we have made) for imaginary numbers. For example,

$$\sqrt{-1}\sqrt{-1}=i\cdot i=i^2=-1;$$

but if we attempted to use the rule of ordinary algebra: $\sqrt{a}\sqrt{b}=\sqrt{ab}$, we should obtain the *incorrect* result

$$\sqrt{-1}\sqrt{-1}=+\sqrt{+1}=1,$$

for we have agreed in this book that $\sqrt{+1} = +\sqrt{+1} = +1$. Avoid the rule $\sqrt{a}\,\sqrt{b} = \sqrt{ab}$ for imaginary numbers, until after very much more thorough study of the subject, by proceeding as directed above.

EXERCISES I: NOTE IX—OPERATIONS ON IMAGINARIES

Perform the operations indicated, where i means $\sqrt{-1}$; when other square roots of negative quantities occur, take $\sqrt{-x} = +\sqrt{x}\,i$ in these exercises:

1. $(\sqrt{-2})^2$.
2. $(-3\,i)^2$.
3. $(2\,i)(3\,i)$.
4. $(3+2\,i)+(2+5\,i)$.
5. $(6-4\,i)-(3+i)$.
6. $(1-i)^2$.
7. $(4+5\,i)(3-2\,i)$.
8. $\dfrac{2-3\,i}{2+3\,i}$.
9. $\dfrac{3}{2\,i}$.
10. $\dfrac{6-i}{2-5\,i}$.
11. $2-4\,i+\dfrac{3}{5-8\,i}$.
12. $3\sqrt{-4}+7\,i$.
13. $2\sqrt{-9}+5\,i-3\sqrt{-8}$.
14. $(2+\sqrt{-9})+(3-2\sqrt{-9})$.
15. $(4+\sqrt{-5})(4-\sqrt{-5})$.
16. $(3+\sqrt{-2})(4-\sqrt{-5})$.
17. $\dfrac{5-\sqrt{-4}}{2+\sqrt{-4}}$.
18. $\dfrac{3+\sqrt{-2}}{2-\sqrt{-2}}$.
19. $\dfrac{1+\sqrt{-3}}{3+\sqrt{-1}}$.
20. $(-\frac{1}{2}+\frac{1}{2}\sqrt{-3})(-\frac{1}{2}-\frac{1}{2}\sqrt{-3})$.
21. $(-\frac{1}{2}+\frac{1}{2}\sqrt{-3})^3$.
22. $(-\frac{1}{2}-\frac{1}{2}\sqrt{-3})^3$.

Write the real and imaginary factors of the following:

23. x^2+y^2.
24. m^2+n^2.
25. $9\,x^2+16\,y^2$.
26. $4\,m^2+n^2$.
27. x^4-y^4.
28. $16\,x^4+y^4$.

37. Quadratic Equations. In solving quadratic equations we found just such numbers as those above. We may now use these answers intelligently; hence, we may regard them as answers. Occasionally, such answers may have some actual meaning in a problem.

Ex. 1. Given $x^2+2=0$, we found (p. 210) $x=\pm\sqrt{-2}$. These answers actually satisfy the equation, for

$$(\pm\sqrt{-2})^2+2=-2+2=0.$$

Ex. 2. Given $x^2-4x+5=0$, we found $x=+2\pm\sqrt{-1}$, or $x=2\pm i$. Substituting these answers for x, we find:

(1) for $x=2+i$:

$$(2+i)^2-4(2+i)+5=(3+4i)-(8+4i)+5=0;$$

(2) for $x=2-i$:

$$(2-i)^2-4(2-i)+5=(3-4i)-(8-4i)+5=0;$$

hence, these answers really satisfy the given equation.

Ex. 3. Given any quadratic $ax^2+bx+c=0$, we found (p. 213)

$$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}.$$

The answers are real, equal, or imaginary, according as b^2-4ac is >0, $=0$, <0. See p. 214. The only difference in our present rules from those of p. 214 is that we now understand how to work with imaginaries, whereas such numbers were then *meaningless*.

EXERCISES II: NOTE IX — QUADRATIC EQUATIONS

Solve the following quadratic equations; check each result:

1. $x^2-2x+2=0$.

2. $x^2-4x+8=0$.

3. $x^2+4x+8=0$.

4. $4x^2-4x+10=0$.

5. $2x^2-3x+5=0$.

6. $x^2-6x+11=0$.

7. $x^2-3x+3=0$.

8. $x^2+x+1=0$.

38. Other Operations. Many other operations in imaginaries are possible, but we shall content ourselves with the preceding after giving a few examples.

Ex. 1. $\sqrt[4]{-1}=\sqrt{\sqrt{-1}}=\sqrt{i}=x+yi.$

Squaring both sides, we have $i=(x^2-y^2)+2xyi$,

whence, $x^2-y^2=0$ and $2xy=1$,

or, $(x-y)(x+y)=0$ and $2xy=1$.

Solving for x and y, we find

$$x = \pm\sqrt{\frac{1}{2}} = \pm\frac{1}{2}\sqrt{2}, \quad or, \quad x = \pm\sqrt{-\frac{1}{2}} = \pm\frac{i}{2}\sqrt{2},$$

but the last result may be discarded since we wish to have x real.

The corresponding values of y are $y = \pm\frac{1}{2}\sqrt{2}$.

It follows that $x + yi = \pm\frac{1}{2}\sqrt{2}\,(1 - i)$. If we consider also the other possibility $\sqrt[4]{-1} = \sqrt{-i}$, we find two other answers:

$$x + yi = \pm\frac{1}{2}\sqrt{2}\,(1 + i).$$

These results may all be checked by raising any one of the results to the fourth power.

Ex. 2. Solve the equation $x^3 - 1 = 0$.

Noting that we can factor the left side,

$$(x - 1)(x^2 + x + 1) = 0,$$

we see that $x = 1$, or else $\qquad x^2 + x + 1 = 0,$

an equation whose roots are $x = -\frac{1}{2} \pm \frac{1}{2}\sqrt{-3}$.

There are therefore *three* answers. (Verify each of them.)

Ex. 3. Solve $x^4 + x^2 + 1 = 0$. Factoring, we find

$$\begin{aligned} x^4 + x^2 + 1 &= (x^4 + 2x^2 + 1) - x^2 = (x^2 + 1)^2 - x^2 \\ &= [(x^2 + 1) - x]\,[(x^2 + 1) + x] \\ &= (x^2 - x + 1)(x^2 + x + 1). \end{aligned}$$

Hence, $x^4 + x^2 + 1 = 0$ gives either

$$x^2 - x + 1 = 0, \qquad \text{or,} \quad x^2 + x + 1 = 0,$$

hence, $\qquad x = \frac{1}{2} \pm \frac{1}{2}\sqrt{-3}, \qquad$ or, $\quad x = -\frac{1}{2} \pm \frac{1}{2}\sqrt{-3}.$

Check: Substitute $x = \frac{1}{2} + \frac{1}{2}\sqrt{-3}$ in $x^4 + x^2 + 1$; we find

$$\left(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{-3}\right)^4 + \left(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{-3}\right)^2 + 1$$
$$= \left(-\tfrac{1}{2} - \tfrac{1}{2}\sqrt{-3}\right) + \left(-\tfrac{1}{2} + \tfrac{1}{2}\sqrt{-3}\right) + 1 = 0.$$

The answers $x = \frac{1}{2} - \frac{1}{2}\sqrt{-3}$, $x = -\frac{1}{2} \pm \frac{1}{2}\sqrt{-3}$ may be verified in an exactly similar way.

No other problems will be solved because the best method — known as De Moivre's theorem — is beyond the scope of this book. Logarithms also lead to imaginary numbers; the discussion of these and other more intricate matters is left to more advanced courses.

NOTE X. SIMULTANEOUS QUADRATICS

39. One Equation Factorable. We have studied in Chapter X pairs of simultaneous equations of which at least one is a quadratic equation. We shall add, in this note, several additional methods.

We saw that any such pair, one of which is linear, may be solved by substitution (p. 255). This method applies also whenever one of the equations can be factored as in the following examples.

Ex. 1. $$\begin{cases} 2x^2 - 3xy + y^2 = 0, & (1) \\ x^2 + y^2 - 10x = 75. & (2) \end{cases}$$

(1) may be written in the form

$$(2x - y)(x - y) = 0, \tag{1}$$

which is equivalent to the two equations

$$(1\,a)\quad 2x - y = 0; \qquad (1\,b)\quad x - y = 0.$$

Solving each of these in combination with (2) by the method of p. 256, we find

from $(1\,a)$: from $(1\,b)$:

$$\begin{cases} x = 5, \\ y = 10, \end{cases} \text{ or } \begin{cases} x = -3, \\ y = -6. \end{cases} \qquad \begin{cases} x = \frac{5}{2} + 5\sqrt{7}, \\ y = \frac{5}{2} + 5\sqrt{7}, \end{cases} \text{ or } \begin{cases} x = \frac{5}{2} - 5\sqrt{7}, \\ y = \frac{5}{2} - 5\sqrt{7}. \end{cases}$$

These answers may all be verified by the student.

These solutions are shown in Fig. 74 by the points marked A, B, C, D, respectively. Equation (2) gives the circle of center (5, 0) and radius 10; equation (1) gives the two straight lines $(1\,a)$ and $(1\,b)$.

This process may be used whenever one *of the equations is of the form*

$$\text{(I)} \qquad Ax^2 + Bxy + Cy^2 = 0,$$

for this equation is always factorable by the methods of §§ 61, 119, pp. 98–225. Such an equation as (I), *i.e.* an equation in which the degree of each term is the same, is called **homogeneous**.

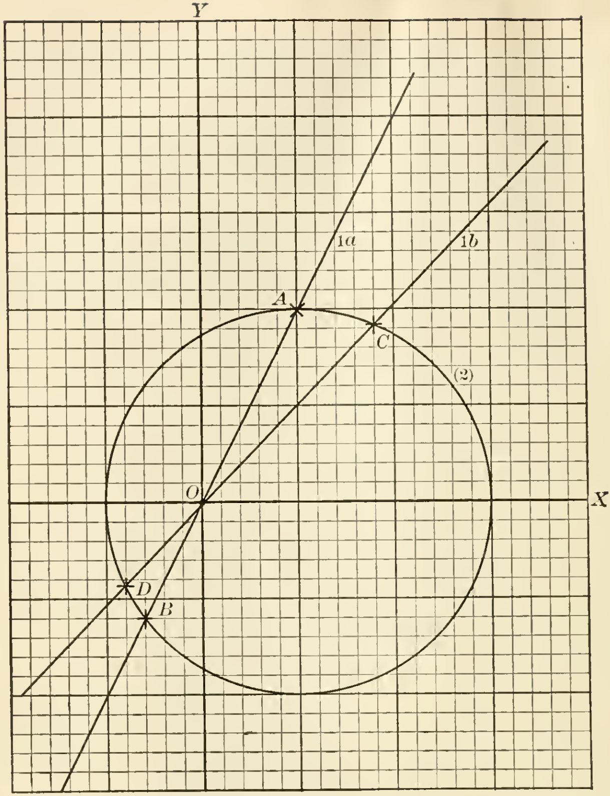


FIG. 74.

A case of another type is illustrated by the example following:

Ex. 2. $\begin{cases} x^2 - y^2 + 2x + 2y = 0, & (1) \\ x^2 - y = 0. & (2) \end{cases}$

Factoring (1), gives $(x - y)(x + y) + 2(x + y) = 0,$

or, $$(x + y)(x - y + 2) = 0,$$

which is equivalent to the two equations:

$$(1\ a) \quad x + y = 0; \qquad (1\ b) \quad x - y + 2 = 0.$$

Solving each of these in combination with (2) by the methods of p. 255, gives

from (1 a):

$$\left\{\begin{matrix} x = 0, \\ y = 0, \end{matrix}\right\} \text{ or } \left\{\begin{matrix} x = -1, \\ y = +1. \end{matrix}\right\}$$

from (1 b):

$$\left\{\begin{matrix} x = -1, \\ y = +1, \end{matrix}\right\} \text{ or } \left\{\begin{matrix} x = +2, \\ y = +4. \end{matrix}\right\}$$

These may be verified by the student. The figure for (1) in Fig. 75 is a pair of lines (1 a) and (1 b); that for (2) is the curve $y = x^2$,

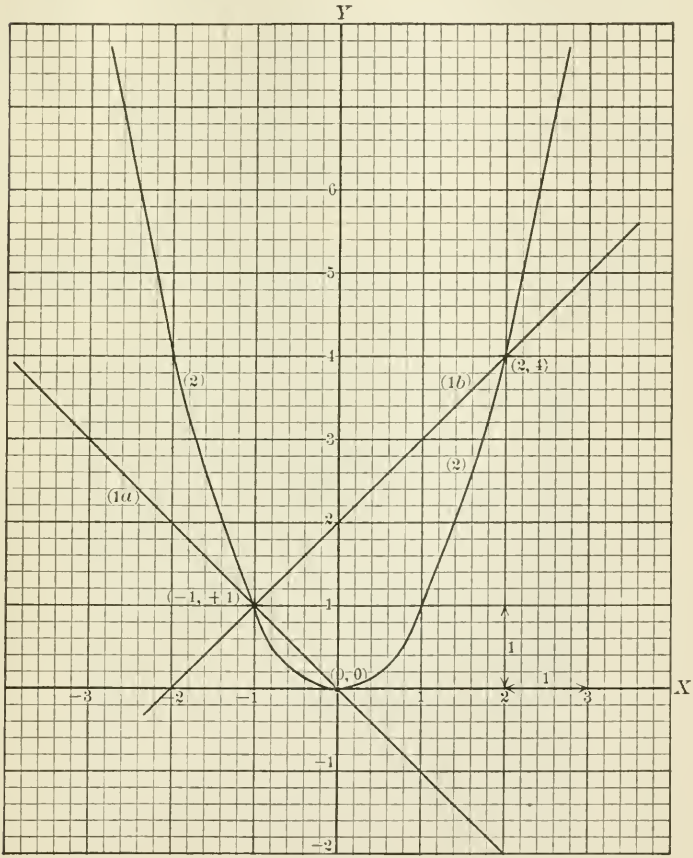


FIG. 75.

which we have drawn before. The point (-1), $(+1)$ appears *twice* as a solution; the figure makes clear *why* this is true.

In general, any quadratic equation containing y^2 may be factored as in Example 2 if an attempted solution for y involves no radicals in x.

Ex. 3. The equation

(1) $$y^2 - 3x^2 - 2xy + 8x - 4y + 3 = 0 \text{ may be written}$$

$$y^2 - 2(x+2)y = 3x^2 - 8x - 3.$$

Solving for y by the usual method (see p. 206), we find

$$y = (x+2) \pm (2x - 1),$$

i.e. (1 *a*) $y = 3x + 1$; or, (1 *b*) $y = -x + 3$.

Equation (1) therefore represents a pair of straight lines (1 *a*) and (1 *b*) (Fig. 76); hence, if (1) occurs as one of a pair of simultaneous quadratics, the pair may be solved as above.

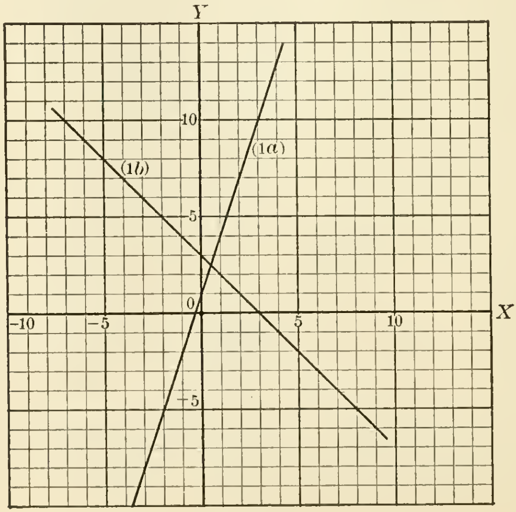


Fig. 76.

40. Type $Ax^2 + Bxy + Cy^2 = D$. If neither of the given pair of quadratics is factorable in the sense of § 39, it is often possible to form a new equation from them which is factorable.

Ex. 1. $\begin{cases} x^2 + 4y^2 = 5, & (1) \\ xy = 1. & (2) \end{cases}$

Multiplying both sides of (1) by -1, both sides of (2) by 5, and adding, we find

(3) $$x^2 - 5xy + 4y^2 = 0,$$

which is factorable (see § 39) and is equivalent to

$$(3\,a) \quad x - 4y = 0, \qquad (3\,b) \quad x - y = 0.$$

Solving each of these with (2), we find

from $(3\,a)$: $\begin{cases} x = 2, \\ y = \frac{1}{2}, \end{cases}$ or $\begin{cases} x = -2, \\ y = -\frac{1}{2}. \end{cases}$ from $(3\,b)$: $\begin{cases} x = 1, \\ y = 1, \end{cases}$ or $\begin{cases} x = -1, \\ y = -1. \end{cases}$

These answers may be verified by the student.

This process may be used whenever *both* equations are of the form

(II) $$Ax^2 + Bxy + Cy^2 = D;$$

the rule is to *eliminate the constant term*, as above, so as to get an equation of the type (I).*

---

* This type of equations also yields to the following method: Let $y = vx$ in both equations; equate the values of x^2 in the resulting equations, and solve for v. Thus, in the preceding example,

(1) becomes $$x^2 + 4v^2x^2 = 5, \text{ or } x^2 = \frac{5}{1 + 4v^2}.$$

(2) becomes $$vx^2 = 1, \quad x^2 = \frac{1}{v}.$$

Equating the values of x^2: $\dfrac{5}{1 + 4v^2} = \dfrac{1}{v}$, or $4v^2 - 5v + 1 = 0$.

Hence, $v = \frac{1}{4}$ or 1, whence $x = \pm 2$ or ± 1, and $y = vx = \pm \frac{1}{2}$ or ± 1, which, when properly paired, give the pairs of solutions found above.

41. Symmetrical Type. Other equations yield to the method of § 40.

Ex. $$\begin{cases} 2xy - x - y + 5 = 0, & (1) \\ x^2 + 6xy + y^2 - 4x - 4y - 5 = 0. & (2) \end{cases}$$

Multiply both sides of (1) by 2; multiply both sides of (2) by -1, then add: (3) $x^2 + 2xy + y^2 - 2x - 2y - 15 = 0$,

or, $(x+y)^2 - 2(x+y) - 15 = 0$,

which is the same as $[(x+y)-5][(x+y)+3] = 0$,

and is therefore equivalent to the two equations

(3 a) $x + y - 5 = 0$, (3 b) $x + y + 3 = 0$.

Solving each of these with (1), we find

from (3 a): $\begin{Bmatrix} x = 5, \\ y = 0, \end{Bmatrix}$ or $\begin{Bmatrix} x = 0, \\ y = 5. \end{Bmatrix}$

from (3 b): $\begin{Bmatrix} x = 1, \\ y = -4, \end{Bmatrix}$ or $\begin{Bmatrix} x = -4, \\ y = 1. \end{Bmatrix}$

This method is successful whenever *both* equations are **symmetrical**; *i.e.* of the form

(III) $$\boldsymbol{Ax^2 + Bxy + Ay^2 + Dx + Dy + F = 0},$$

so that an interchange of x and y does not affect the equation; the rule is to make the quadratic part of the new equation a perfect square, *by multiplying each equation by the value of $B - 2A$ in the other equation, and then subtracting;* this is practically what we did above.*

42. General Method of Inspection. It can be shown that any pair of simultaneous quadratic equations can be

---

* This type (both equations symmetrical) can be solved also by making the substitution $x = u + v$, $y = u - v$; in the example above this gives

from (1) $2u^2 - 2v^2 - 2u + 5 = 0$,
from (2) $8u^2 - 4v^2 - 8u - 5 = 0$.

Adding these, after multiplying the first by -2, we get

$$4u^2 - 4u - 15 = 0,$$

which gives $u = \frac{5}{2}$ or $-\frac{3}{2}$; hence, $v^2 = \frac{1}{4}(8u^2 - 8u - 5) = \frac{25}{4}$ and $v = \pm\frac{5}{2}$, whence, $x = u + v = 5$ or 0, or 1 or -4, and $y = u - v = 0$ or 5, or -4 or 1, which give the preceding solutions when properly paired.

solved in a similar manner if the proper multipliers can be found. Thus, on multiplying both sides of the first by λ and adding it to the second, there are always *three* values of λ which make the new equation factorable in the sense of § 39. In many examples, however, it may be impossible for the student at present *to find these values of* λ.

Ex. 1. $\begin{cases} x^2 - xy + y^2 + 2x - y = 3, & (1) \\ x^2 + y^2 + 4x - 2y = 5. & (2) \end{cases}$

Choose $\lambda = -2$; *i.e.* multiply (1) by -2 and then add equation (2),

(3) $$-x^2 + 2xy - y^2 = -1,$$

or, changing signs and transposing, we have

$$(x - y)^2 - 1 = 0,$$

whence, (3) is equivalent to the two equations

(3 a) $x - y + 1 = 0,$ (3 b) $x - y - 1 = 0.$

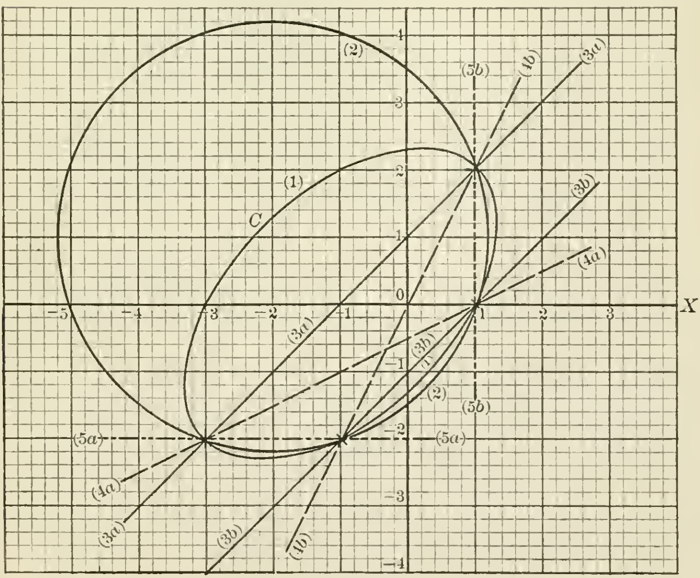


FIG. 77.

Combining each of these with (2), we find the solutions (Fig. 77)

from $(3\,a)$:

$$\begin{Bmatrix} x=1, \\ y=2, \end{Bmatrix} \text{ or } \begin{Bmatrix} x=-3, \\ y=-2, \end{Bmatrix}$$

from $(3\,b)$:

$$\begin{Bmatrix} x=1, \\ y=0, \end{Bmatrix} \text{ or } \begin{Bmatrix} x=-1, \\ y=-2, \end{Bmatrix}$$

which the student may verify directly. The figure shows the picture for each of the equations used, and also those which follow.

The choice $= -1$ is equally successful (Fig. 77, $5\,a$ and $5\,b$).

The choice $= -\frac{5}{3}$ is equally successful (Fig. 77, $4\,a$ and $4\,b$).

As a general method this is undoubtedly superior; its details are too lengthy for discussion here. Notice all the methods given above are special cases of this one.

EXERCISES I: NOTE X — SIMULTANEOUS QUADRATICS

Solve each of the following pairs of equations and draw a figure showing the curves and their points of intersection:

1. $\begin{cases} x^2 - y^2 = 0, \\ x^2 - 2\,y = 0. \end{cases}$

2. $\begin{cases} x^2 - 5\,xy + 6\,y^2 = 0, \\ x^2 + y^2 = 100. \end{cases}$

3. $\begin{cases} x^2 - 5\,xy + 4\,y^2 = 0, \\ xy = 36. \end{cases}$

4. $\begin{cases} x^2 - 2\,xy + y^2 - 3(x-y) = 0, \\ y = x^2. \end{cases}$

5. $\begin{cases} y^2 - 3\,x^2 - 2\,xy + 8\,x - 4\,y + 3 = 0 \text{ (see p. 400)}, \\ y = x^2. \end{cases}$

6. $\begin{cases} x^2 - y^2 = 33, \\ xy + y^2 = 44. \end{cases}$

7. $\begin{cases} x^2 + 3\,xy + y^2 + 2\,x + 2\,y = 17, \\ x^2 + xy + y^2 = 7. \end{cases}$

8. $\begin{cases} x^2 + xy + y^2 = 21, \\ x^2 - xy + y^2 - 2\,x - 2\,y = 23. \end{cases}$

9. $\begin{cases} x^2 - xy + y^2 = 3, \\ x^2 + xy + y^2 = 7. \end{cases}$

10. $\begin{cases} x^2 + 4\,xy + y^2 + 4\,x + 4\,y = 89, \\ x^2 - 2\,xy + y^2 - 2\,x - 2\,y = 11. \end{cases}$

11. $\begin{cases} x^2 - 2\,xy + y^2 + 4\,x + 4\,y = 16, \\ 2\,x^2 + xy + 2\,y^2 - 2\,x - 2\,y = 7. \end{cases}$

SUMMARY OF APPENDIX, pp. 354-404

TABLES

[The student doubtless knows many of these; some others should be learned, stress being laid on the tables in the metric system.]

I. TABLE OF SIGNS

$+$, read *plus*. $-$, read *minus*.

$\times$, or $\cdot$, read *times*, *multiplied by*, *multiplied into*, or *into* $(a \times b = a \cdot b = ab)$.

$\div$, :, or /, read *divided by* or *over*.

$=$, read *equals*, or *is equal to*.

$\neq$, read *is not equal to*.

$>$, read *is greater than*. $\geqq$, read *is greater than or equal to*.

$<$, read *is less than*. $\leqq$, read *is less than or equal to*.

[The signs just written may be easily remembered by noting that the opening faces the greater quantity.]

a^2, read *a square* $= a \times a$. a^3, read *a cube* $= a \times a \times a$.

a^n, read *a to the nth power*, *a to the power n*, or *a with an exponent* $n = a \times a \cdots$ to n factors.

( ), [ ], { }, $\overline{\quad}$, called signs of aggregation; a general term used for them is *parentheses*, and quantities inclosed by them are read *the quantity* $\cdots$ or *the expression* $\cdots$. To distinguish them, we call ( ) parentheses; [ ] brackets, { } braces; $\overline{\quad}$ the vinculum.

$\sqrt{a}$, read *the square root of a*; this is also written $a^{\frac{1}{2}}$ and read *a to the power* $\frac{1}{2}$, see pp. 192, 285.

$\sqrt[3]{a}$, read *the cube root of a*; this is also written $a^{\frac{1}{3}}$ and read *a to the power* $\frac{1}{3}$, see pp. 192, 285.

$\sqrt[n]{a}$, read *the nth root of a*; this is also written $a^{\frac{1}{n}}$, and read *a to the power* $\frac{1}{n}$, see pp. 192, 285.

II. TABLES OF WEIGHTS AND MEASURES

[The customary abbreviation of each measure is indicated. The most important units are printed in black-faced type.]

A. Measures of Quantity

1. *Table of Measures of Common Objects*

12 units	= 1 dozen (doz.).
20 units	= 1 score.
12 dozen	= 1 gross (gr.).
12 gross	= 1 great gross (G. gr.).

2. *Table of Measures of Paper*

24 sheets	= 1 quire.
20 quires	= 1 ream.
2 reams	= 1 bundle.
5 bundles	= 1 bale.

B. Measures of Value

1. *Table of American Money*

10 cents (¢) or (ct.)	= 1 dime.
10 dimes	= 1 **dollar** ($)
10 dollars	= 1 eagle.

Other measures are the quarter (= 25 cents), the half-dollar (= 50 cents), and the mill (= $\frac{1}{10}$ cent).

2. *Table of English Money*

4 farthings (far.)	= 1 penny (*d.*).
12 pence	= 1 **shilling** (*s.*).
20 shillings	= 1 **pound** (£).

Other measures are the crown (= 5 shillings) and the guinea (= 21 shillings).

3. *Table of French Money*

10 millimes	= 1 centime.
10 centimes	= 1 decime.
10 decimes	= 1 **franc** (fr.).

4. *Table of German Money*

100 pfennigs (pf.) = 1 **mark** (mk.).

5. *Table of Italian Money*

100 centisimos = 1 **lira** (lr.).

In the following tables of equivalents, *three significant figures* are given, except in a few important instances. The student should realize that it is *not* the number of decimal places, but rather the number of significant figures, which determines the degree of accuracy.

6. *Table of Equivalents*

1*d.*	= $0.0203.	1 fr.	= $0.193.
1*s.*	= $0.243.	1 mk.	= $0.238.
1 £	**= $4.8665.**	1 lr.	= $0.193.

C. Measures of Length

1. *Table of English Measure of Length*

12 inches (in.)	= 1 **foot** (ft.).
3 feet	= 1 yard (yd.).
$5\frac{1}{2}$ yards	= 1 rod (rd.).
320 rods	= 1 **mile** (mi.).

2. *Table for Metric Measure of Length*

10 millimeters (mm.)	= 1 centimeter (cm.).
10 centimeters	= 1 decimeter (dm.).
10 decimeters	= 1 meter (m.).
10 meters	= 1 dekameter (Dm.).
10 dekameters	= 1 hektometer (Hm.).
10 hektometers	= 1 kilometer (Km.).
10 kilometers	= 1 myriameter (Mm.).

The *meter* is approximately one four-millionth of the circumference of the earth, measured along a meridian through Dunkirk, France.

3. *Table of Equivalents*

ENGLISH TO METRIC	METRIC TO ENGLISH
1 in. = 2.54 cm.	1 cm. = 0.3937 in.
1 ft. = 30.5 cm.	1 m. = 39.4 in. = 3.28 ft.
1 yd. = 91.4 cm. = .914 m.	1 Km. = 0.621 mi.
1 mi. = 1.61 Km.	

D. Measures of Area

1. *Table of English Measure of Area*

144 square inches (sq. in.)	= 1 square foot (sq. ft.).
9 square feet	= 1 square yard (sq. yd.).
$30\frac{1}{4}$ square yards	= 1 square rod (sq. rd.).
160 square rods	= 1 acre (A.).
640 acres	= 1 square mile (sq. mi.).

2. *Table of Metric Measures of Area*

100 square millimeters (sq. mi.)	= 1 square centimeter (sq. cm.).
100 square centimeters	= 1 square decimeter (sq. dm.).
100 square decimeters	= 1 square meter.
100 square meters	= 1 square dekameter (sq. Dm.) or are (a.).
100 square dekameters	= 1 square hektometer (sq. Hm.) or hektare.
100 square hektometers	= 1 square kilometer (sq. Km.).

3. *Table of Equivalents*

ENGLISH TO METRIC	METRIC TO ENGLISH
1 sq. in. = 6.45 sq. cm.	1 sq. cm. = .155 sq. in.
1 sq. yd. = 0.835 sq. m.	1 sq. m. = 1.20 sq. yd.
1 sq. rd. = .253 a.	1 a. = 3.95 sq. rd.

E. Measures of Volume

1. *Table of English Measure of Volume*

1728 cubic inches (cu. in.)	= 1 cubic foot (cu. ft.).
27 cubic feet	= 1 cubic yard (cu. yd.).
128 cubic feet	= 1 cord (of wood) (cd.).

2. *Table of Metric Measurements of Volume*

1000 cubic millimeters (cu. mm.)	= 1 cubic centimeter (cu. cm.).
1000 cubic centimeters	= 1 cubic decimeter (cu. dm.).
1000 cubic decimeters	= 1 cubic meter (cu. m.) or 1 stere (of wood) (st.).

3. *Table of Equivalents*

ENGLISH TO METRIC	METRIC TO ENGLISH
1 cu. in. = 16.4 cu. cm.	1 cu. cm. = 0.0612 cu. in.
1 cu. yd. = 0.76 cu. m.	1 cu. m. = 1.31 cu. yd.
1 cd. = 3.6 st.	1 st. = 0.28 cd.

F. Measures of Time

60 seconds (sec.)	= 1 minute (min.).
60 minutes	= 1 hour (hr.).
24 hours	= 1 day (da.).
7 days	= 1 week (wk.).
14 days	= 1 fortnight.
12 months	= 1 year (yr.).
365 days	= 1 year.
366 days	= 1 leap year.
10 years	= 1 decade.
100 years	= 1 century.

The number of days in the different months vary. February has 28 days, except for leap year when the extra day is added to it, giving it 29. September, April, June, and November each have 30 days. The other months each have 31 days. Every year divisible by four, except centennial years not divisible by 400, is a leap year.

G. Measures of Capacity

1. *Tables of English Measurement of Capacity*

DRY MEASURE		LIQUID MEASURE	
2 pints (pt.)	= 1 quart (qt.).	4 gills (gi.)	= 1 pint (pt.).
8 quarts	= 1 peck (pk.).	2 pints	= 1 quart (qt.).
4 pecks	= 1 bushel (bu.).	4 quarts	= 1 gallon (gal.).

Other measures are the barrel (bbl. = $31\frac{1}{2}$ gal.) and the hogshead (hhd. = 2 bbl.).

The gallon contains 231 cu. in.

2. *Tables of Metric Measurement of Capacity*

10 milliliters (ml.)	= 1 centiliter (cl.).
10 centiliters	= 1 deciliter (dl.).
10 deciliters	= 1 liter (l.).
10 liters	= 1 dekaliter (Dl.).
10 dekaliters	= 1 hectoliter (Hl.).
10 hectoliters	= 1 kiloliter (Kl.).

The liter contains 1 cu. dm. or 61.02 cu. in.

3. *Table of Equivalents*

ENGLISH TO METRIC		METRIC TO ENGLISH	
1 dry qt.	= 1.10 l.	1 l.	= 0.908 dry qt.
1 liquid qt.	= 0.947 l.	1 l.	= 1.0567 liquid qt.
1 bu.	= 35.2 l.	1 Hl.	= 2.84 bu.
1 gal.	= 3.79 l.	1 l.	= 0.264 gal.

H. Measures of Weight

1. *Table of Avoirdupois Weight*

16 drams (dr.).	= 1 ounce (oz.).
16 ounces	= 1 pound (lb.).
100 pounds	= 1 hundredweight (cwt.).
112 pounds	= 1 long (or English) hundredweight.
20 hundredweight	= 1 ton (T).
20 long hundredweight	= 1 long (or English) ton.
7000 grains	= 1 pound.

2. *Table of Troy Weight*

24 grains (gr.)	= 1 pennyweight (pwt.).
20 pennyweights	= 1 ounce (oz.).
12 ounces	= 1 pound (lb).

3. *Table of Apothecaries' Weight*

20 grains (gr.)	= 1 scruple (℈).
3 scruples	= 1 dram (ʒ).
8 drams	= 1 ounce (℥).
12 ounces	= 1 pound (lb.).

4. *Table of Metric Measure of Weight*

10 milligrams (mg.)	= 1 centigram (cg.).
10 centigrams	= 1 decigram (dg.).
10 decigrams	= 1 gram (g.).
10 grams	= 1 dekagram (Dg.).
10 dekagrams	= 1 hektogram (Hg.).
10 hektograms	= 1 kilogram (Kg.).
1000 kilograms	= 1 metric ton (T.).

Table of Equivalents

English to Metric

1 oz. Troy	= 31.1 g.
1 oz. Av.	= 28.4 g.
1 lb. Av.	= 0.454 kg.
1 ton	= 0.907 metric ton.

Metric to English

1 g.	= 0.0322 oz. Troy.
1 g.	= 0.0353 oz. Av.
1 Kg.	= 2.2046 lb. Av.
1 metric ton	= 1.10 tons.

I. Measurement of Angles

60 seconds ($''$)	= 1 minute ($'$).
60 minutes	= 1 degree ($^\circ$).
90 degrees	= 1 right angle.
360 degrees	= 1 perigon (circumference).

$\pi = 3.14159265\cdots$.

$\;\;= 3.1416\cdots$ (nearly).

$\;\;= 3\frac{1}{7}$ (roughly).

$$1 \text{ radian} = \frac{180^\circ}{\pi} = \frac{180^\circ}{3.1416\cdots} = 57^\circ\ 17'\ 45'' \text{ (nearly)}.$$

$$1^\circ = \frac{3.1416\cdots}{180} \text{ radians} = 0.0175\cdots \text{ radians}.$$

J. Measurement of Temperature

The unit of temperature is the degree. The Centigrade thermometer is graduated so that the temperature at which water freezes is 0°, and the temperature at which it boils is 100°.

The Fahrenheit thermometer is graduated so that the temperature at which water freezes is 32°, and the temperature at which it boils is 212°.

The Réaumur thermometer is graduated so that the temperature at which water freezes is 0°, and the temperature at which it boils is 80°.

If C denotes the Centigrade record of temperature and F the Fahrenheit, then,

$$C = \tfrac{5}{9}(F - 32°), \quad F = \tfrac{9}{5}C + 32°.$$ (See p. 143.)

If R denotes the Réaumur temperature,

$$R = \tfrac{4}{5}C = \tfrac{4}{9}(F - 32°).$$

A comparison of these three scales of temperature may be readily seen from the following table:

	Centigrade	Fahrenheit	Réaumur
Freezing	0	32	0
Boiling	100	212	80
From boiling to freezing . .	100	180	80

III. TABLES OF SIMPLE FORMULAS FROM PHYSICS

1. *Falling Bodies.*

With the notation s = space passed over (in ft. or cm.), t = time (in sec.), v = velocity (in ft. per sec. or cm. per sec.). g = gravitational acceleration = 32.16 ft. per sec. per sec. = 981 cm. per sec. per sec.

If dropped from rest:

$$v = gt, \quad s = \tfrac{1}{2}gt^2, \quad v^2 = 2gs.$$

In general, $\quad v - v_0 = gt, \quad s - s_0 = \tfrac{1}{2}gt^2 + v_0t,$

if v_0 = initial velocity, s_0 = initial distance.

2. *Lever, or Balance.*

$$dW = Dw.$$ See Fig. 30, p. 176.

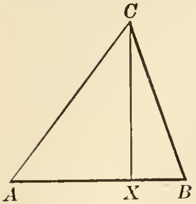
3. *Boyle's Law: Pressure and Volume.*

$$pv = \text{constant}.$$ See p. 218.

Other formulas given as needed in the text.

IV. TABLES OF GEOMETRIC MENSURATION FORMULAS

Triangle

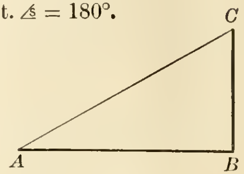


TRIANGLE

Area $= \frac{1}{2}$ (base $\times$ altitude) $= \frac{1}{2}\, AB \cdot CX$
$= \frac{1}{2}\, b \cdot a$, where b = base $= AB$, and
a = altitude $= CX$.

Sum of angles $= \angle A + \angle B + \angle C$
$= 2$ rt. $\angle$s $= 180°$.

Right Triangle



RIGHT TRIANGLE

$\angle B =$ rt. $\angle = 90°$,
$\angle A + \angle C = 90°$.

(Hypotenuse)^2 = sum of squares of the perpendicular sides:

$$AC^2 = AB^2 + BC^2.$$

If $AC = h$ (= hypotenuse), $AB = b$ (= base), $CB = a$ (= altitude), the formula becomes, $$h^2 = a^2 + b^2.$$

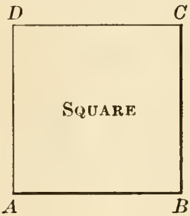
Ratios in Right Triangle

If $\frac{a}{h} = s$ (= sine of $\angle A$), $\frac{b}{h} = c$ (= cosine of $\angle A$),

and $\frac{a}{b} = t$ (= tangent of $\angle A$), then:

$$\frac{s}{c} = t \qquad s^2 + c^2 = 1$$

$$1 + \frac{1}{c^2} = t^2 \qquad 1 + \frac{1}{s^2} = \frac{1}{t^2}.$$

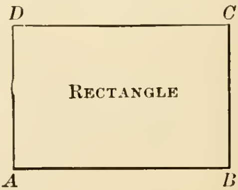


SQUARE

Square

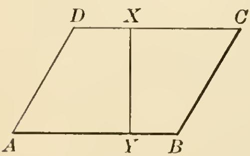
Area = (length of side)$^2 = AB^2 = s^2$, where $s = AB$ = length of side.

Rectangle



RECTANGLE

Area = base $\times$ altitude $= AB \times BC$
$= b \cdot a$, where b = base $= AB$,
and a = altitude $= BC$.



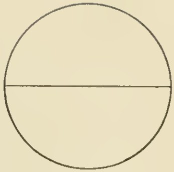
PARALLELOGRAM

Parallelogram

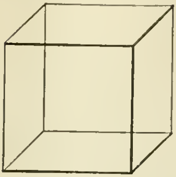
Area = base $\times$ altitude $= AB \times XY$
$= b \cdot a$, where b = base $= AB$, and
where a = altitude $= XY$.

Circle

Circumference $= 2\,\pi \times$ radius $= 2\,\pi r$;
Area $= \pi \times (\text{radius})^2 = \pi r^2$,
where $r =$ radius of circle,
and, see p. 412, $\pi = 3.1416 \cdots = 3\frac{1}{7}$ (nearly).



CIRCLE



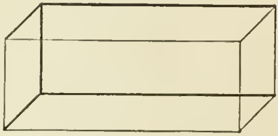
CUBE

Cube

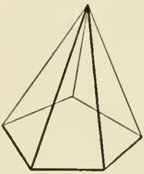
Volume $= (\text{length of side})^3 = s^3$,
where $s =$ length of side.

Rectangular Parallelopiped

Volume =
(area of base) × altitude.



RECTANGULAR PARALLELOPIPED



PYRAMID

Pyramid — any Base

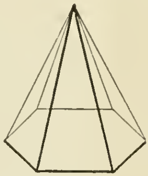
Volume =
$\frac{1}{3}$ (area of base) × altitude.

Regular Pyramid

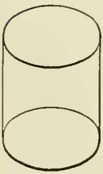
(Base a regular polygon; perpendicular from vertex meets base at center.)

Lateral area $= \frac{1}{2}$ (perimeter of base) × (altitude of face).

The altitude of face is the altitude of any one of the triangular faces, often called *slant height*.



REGULAR PYRAMID



RIGHT CYLINDER

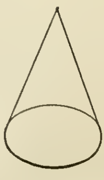
Right Cylinder

Lateral area = (circumference of base) × altitude.
Volume = (area of base) × altitude.

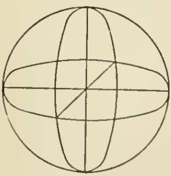
Right Cone

Lateral area $= \frac{1}{2}$ (circumference of base) × slant height (*i.e.* length of element).

Volume $= \frac{1}{3}$ (area of base) × altitude.



RIGHT CONE



SPHERE

Sphere

Area $= 4\,\pi \times (\text{radius})^2 = 4\,\pi r^2$.
Volume $= \frac{4}{3}\,\pi \times (\text{radius})^3 = \frac{4}{3}\,\pi r^3$.

INDEX

(The numbers refer to pages. v. = *vide*.)